Analytic families of eigenfunctions on a reductive symmetric space

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Abstract

The asymptotic behavior of holomorphic families of generalized eigenfunctions on a reductive symmetric space is studied. The family parameter is a complex character on the split component of a parabolic subgroup. The main result asserts that the family vanishes if a particular asymptotic coefficient does. This allows an induction of relations between families that will be applied in forthcoming work on the Plancherel and the Paley-Wiener theorem.

Contents

Introduction

In harmonic analysis on a reductive symmetric space X an important role is played by families of generalized eigenfunctions for the algebra $\mathbb{D}(\mathrm{X})$ of invariant differential operators. Such families arise for instance as matrix coefficients of representations that come in series, such as the (generalized) principal series. In particular, relations between such families are of great interest. We recall that a real reductive group G, equipped with the left times right multiplication action, is a reductive symmetric space. In the case of the group, examples of the mentioned relations are functional equations for Eisenstein integrals, see [23] and [25], or Arthur-Campoli relations for Eisenstein integrals, see [1], [14]. In this paper we develop a general tool to establish relations of this kind. We show that they can be derived from similar relations satisfied by the family of functions obtained by taking one particular coefficient in a certain asymptotic expansion. Since the functions in the family so obtained are eigenfunctions on symmetric spaces of lower split rank, this yields a powerful inductive method; we call it *induction of relations*. In the case of the group, a closely related lifting theorem by Casselman was used by Arthur in the proof of the Paley-Wiener theorem, see [1], Thm. II.4.1. However, no proof seems yet to have appeared of Casselman's theorem.

The tools developed in this paper are used in [11], and they will also be applied in the forthcoming papers [12] and [13]. For example, it is the induction of relations that allows us to establish symmetry properties of certain integral kernels appearing in a Fourier inversion formula in [11]. Also in [11], the induction of relations is used to define generalized Eisenstein integrals corresponding to non-minimal principal series. In [12], the results of this paper will be applied to identify these 'formal' Eisenstein integrals with those defined in Delorme [18]. This is a key step towards the Plancherel decomposition. The results will also be applied to establish functional equations for the Eisenstein integrals. Applied in this manner our technique serves as a replacement for the use of the Maass-Selberg relations as in Harish-Chandra [25] and [18]. On the other hand, in [13] we apply our tool to show that Arthur-Campoli relations satisfied by normalized Eisenstein integrals of spaces of lower split rank induce similar relations for normalized Eisenstein integrals of X. This result is then used to prove a Paley-Wiener theorem for X that generalizes Arthur's theorem for the group. In

particular, the missing proof of Casselman's theorem will then be circumvented by means of a technique of the present paper.

It should be mentioned that in the case of the group, induction of Arthur-Campoli relations for unnormalized Eisenstein integrals is easily derived from their integral representations (see [1], p. 77, proof of Lemma 2.3). For normalized Eisenstein integrals, which are not representable by integrals, the result seems to be much deeper, also in the group case.

One of the interesting features of the theory is that it also deals with families of functions that are not necessarily globally defined on the space X but on a suitable open dense subset.

Asymptotic behavior of eigenfunctions on a symmetric space has been studied at many other places in the literature. The following papers hold results that are related to some of the ideas of the present paper [22], [20], [32], [24], [25], [26], [28], [30], [17], [33], [1], [29], [6], [15].

The core results of this paper were found and announced in the fall of 1995, when both authors were guests at the Mittag-Leffler Institute. In the same period Delorme announced his proof of the Plancherel theorem, which has now appeared in [19].

We shall now explain the contents of this paper in more detail. The space X is of the form G/H, with G a real reductive Lie group of Harish-Chandra's class, and H an open subgroup of the set of fixed points for an involution σ of G.

The group G has a σ-stable maximal compact subgroup K, let θ be the associated Cartan involution of G. Let $P_0 = M_0 A_0 N_0$ be a fixed minimal $\sigma \circ \theta$-invariant parabolic subgroup of G, with the indicated Langlands decomposition. The Lie algebra $\mathfrak{a}_0$ of A_0 is invariant under the infinitesimal involution σ; we denote the associated -1 eigenspace in $\mathfrak{a}_0$ by $\mathfrak{a}_{\mathrm{q}}$. Its dimension is called the split rank of X. Let A_{q} be the vectorial subgroup of G with Lie algebra $\mathfrak{a}_{\mathrm{q}}$ and let $A_{\mathrm{q}}^{\mathrm{reg}}$ be the set of regular points relative to the adjoint action of A_{q} in $\mathfrak{g}$. Then $\mathrm{X}_+ := K A_{\mathrm{q}}^{\mathrm{reg}} H$ is a K-invariant dense open subset of X. Let A_{q}^+ be the open chamber in A_{q} determined by P_0. Then X_+ is a finite union of disjoint sets of the form $K A_{\mathrm{q}}^+ v H$, with v in the normalizer of $\mathfrak{a}_{\mathrm{q}}$ in K. In this introduction we assume, for simplicity of exposition, that $\mathrm{X}_+ = K A_{\mathrm{q}}^+ H$. This assumption is actually fulfilled in the case that X is a group.

Let (τ, V_τ) be a finite dimensional continuous representation of K. Then by $C^\infty(\mathrm{X}_+ : \tau)$ we denote the space of smooth functions $f \colon \mathrm{X}_+ \to V_\tau$ that are τ-spherical, i.e., $f(kx) = \tau(k) f(x)$, for all $x \in \mathrm{X}_+$ and $k \in K$.

Let $\mathcal{P}_\sigma$ denote the (finite) set of $\sigma \circ \theta$-invariant parabolic subgroups of G containing A_{q}. Let $Q = M_Q A_Q N_Q$ be an element of $\mathcal{P}_\sigma$. Then σ restricts to an involution of $\mathfrak{a}_Q$, the Lie algebra of A_Q; we denote its -1 eigenspace by $\mathfrak{a}_{Q\mathrm{q}}$. In the first part of the paper we study a family f of the following type (cf. Definition 7.1). The family is a smooth map of the form

$$f \colon \Omega \times \mathrm{X}_+ \to V_\tau,$$

with Ω an open subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$, the complexified linear dual of $\mathfrak{a}_{Q\mathrm{q}}$. It is assumed that f is holomorphic in its first variable. Moreover, for every $\lambda \in \Omega$ the

function $f_\lambda := f(\lambda, \cdot\,)$ belongs to $C^\infty(\mathrm{X}_+ : \tau)$. It is furthermore assumed that the functions f_λ allow suitable exponential polynomial expansions along A_q^+. More precisely, we assume, for $m \in M_0$ and $a \in A_\mathrm{q}^+$, that

$$f_\lambda(ma) = \sum_{s \in W/W_Q} a^{s\lambda - \rho_{P_0}} \sum_{\xi \in -sW_QY + \mathbb{N}\Sigma(P_0)} a^{-\xi} q_{s,\xi}(\lambda, \log a, m). \qquad (0.1)$$

Here W is the Weyl group of $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q})$ and W_Q is the centralizer of $\mathfrak{a}_{Q\mathrm{q}}$ in W. Moreover, $\Sigma(P_0)$ denotes the collection of roots from Σ occurring in N_0 and Y is a finite subset of ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the annihilator of $\mathfrak{a}_{Q\mathrm{q}}$ in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Finally, the $q_{s,\xi}$ are smooth functions, holomorphic in the first and polynomial in the second variable. Thus, we impose a limitation on the set of exponents and assume that the coefficients depend holomorphically on the parameter λ. The type of convergence that we impose on the expansion (0.1) is described in general terms in the preliminary Section 1.

We show that the functions f_λ actually allow exponential polynomial expansions similar to (0.1) along any (possibly non-minimal) $P \in \mathcal{P}_\sigma$. These expansions are investigated in detail in Sections 3 and 7. Their coefficients are families of $\tau|_{M_P \cap K}$-spherical functions on $\mathrm{X}_{P,+}$, the analogue of X_+ for the lower split rank symmetric space $\mathrm{X}_P := M_P/M_P \cap H$.

The operators from $\mathbb{D}(\mathrm{X})$ do also allow expansions along every $P \in \mathcal{P}_\sigma$. In Section 4 this is shown by investigating a radial decomposition that reflects the decomposition $G = KM_PA_{P\mathrm{q}}H$. It is of importance that the coefficients in these expansions are globally defined smooth functions on M_P, see Prop. 4.10 and Cor. 4.9. From the expansions we derive that the algebra $\mathbb{D}(\mathrm{X})$ acts on the space of families of the above type, see Prop. 7.6.

In Section 8 we introduce the notion of asymptotic s-globality of a family along P. Losely speaking, it means that the coefficients $q_{s,\xi}(\lambda, \log a, \cdot\,)$ of the expansion along P extend smoothly from $\mathrm{X}_{P,+}$ to the full space X_P, for every $\xi \in (sW_QY - \mathbb{N}\Sigma(P))|_{\mathfrak{a}_{P\mathrm{q}}}$. This notion is proved to be stable under the action of $\mathbb{D}(\mathrm{X})$.

In Section 9 we impose three other conditions on the family. The first is that each member satisfies a system of differential equations of the form

$$Df_\lambda = 0 \qquad (D \in I_{\delta,\lambda}).$$

Here $I_{\delta,\lambda}$ is a certain cofinite ideal in the algebra $\mathbb{D}(\mathrm{X})$ depending polynomially on $\lambda \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ in a suitable way. Accordingly, λ is called the spectral parameter of the family. The second condition imposed is a suitable condition of asymptotic globality along certain parabolic subgroups P with $\dim(\mathfrak{a}_\mathrm{q}/\mathfrak{a}_{P\mathrm{q}}) = 1$. Thirdly, it is required that the domain Ω for the parameter λ is unbounded in certain directions (see Defn. 9.9).

The first main result of the paper is then the following vanishing theorem, see Theorem 9.10.

The vanishing theorem. *Let f be a family as above, and assume that the coefficient of $\lambda - \rho_Q$ in the expansion along Q vanishes for λ in a non-empty open subset of Ω. Then the family f is identically zero.*

In the proof the globality assumption is needed to link suitably many asymptotic coefficients together; the vanishing of one of them then inductively causes the vanishing of others. In the induction step a key role is played by the observation that a symmetric space cannot have a continuum of discrete series.

The importance of the vanishing theorem is that it applies to many families that naturally arise in representation theory. In the present paper we show that this is so for Eisenstein integrals associated with the minimal principal series for X; in [12] we will show that Eisenstein integrals obtained by parabolic induction from discrete series form a family of the above type. The idea is that the latter Eisenstein integrals can be obtained from those associated with the minimal principal series by the application of residual operators with respect to the spectral parameter. Such residual operators occur in our papers [10] and [11].

A suitable class of operators containing the residual operators is formed by the Laurent operators. In the second half of the paper we study the application of them to suitable families of eigenfunctions, with respect to the spectral parameter. The Laurent operators are best described by means of Laurent functionals, see Sections 10 and 11.

In Section 12 we introduce a special type of families g of eigenfunctions. It is of the above type, with Ω dense in $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, P a minimal parabolic subgroup in $\mathcal{P}_\sigma$, and satisfies some additional requirements, see Definition 12.8. One of these is that the family and its asymptotic expansions should depend meromorphically on the spectral parameter $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with singularities along translated root hyperplanes. This allows the application of Laurent functionals with respect to the spectral parameter. More precisely, let $Q \in \mathcal{P}_\sigma$ contain P, and let $\mathcal{L}$ be a Laurent functional on ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. From the family g a new family $f = \mathcal{L}_* g$, with a spectral parameter from $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, is obtained by the application of $\mathcal{L}$ to the ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$-component of the spectral parameter. In Theorem 13.12 it is shown that the resulting family $\mathcal{L}_* g$ satisfies the requirements of the vanishing theorem, provided the special family g satisfies certain holomorphic asymptotic globality conditions.

In Section 14 we introduce partial Eisenstein integrals associated with a minimal parabolic subgroup P from $\mathcal{P}_\sigma$. The partial Eisenstein integrals are spherical generalized eigenfunctions on X_+ obtained from the normalized Eisenstein integral $E^\circ(P\colon \lambda)$, ($\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ generic), by splitting it according to its exponential polynomial expansion along P. More precisely, the exponents of $E^\circ(P\colon \lambda)$ are contained in $W\lambda - \rho_P - \mathbb{N}\Sigma(P)$; the partial Eisenstein integrals $E_{+,s}(P\colon \lambda)$, for $s \in W$, are the smooth spherical functions on X_+ determined by the requirements that

$$E^\circ(P\colon \lambda) = \sum_{s\in W} E_{+,s}(P\colon \lambda)$$

and the set of exponents of $E_{+,s}(P\colon \lambda)$ along P should be contained in $s\lambda - \rho_P - \mathbb{N}\Sigma(P)$. It is then shown that the partial Eisenstein integrals yield examples of the special families mentioned above. Moreover, if $Q \in \mathcal{P}_\sigma$, $Q \supset P$, let W^Q be the collection of minimal length (with respect to $\Sigma(P)$) coset representatives for

W/W_Q in W. Then it is shown that for each $t \in W_Q$ the family

$$f_t = \sum_{s \in W^Q} E_{+,st}(P : \cdot) \tag{0.2}$$

satisfies the additional holomorphic asympotic globality property guaranteeing that $\mathcal{L}_* f_t$ satisfies the hypothesis of the vanishing theorem, for $\mathcal{L}$ a Laurent functional on ${}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$.

In Section 15 the asymptotic behavior of $\mathcal{L}_* f_t$ is investigated, and the coefficient of $a^{\lambda - \rho_Q}$ in the expansion along Q is expressed in terms of partial Eisenstein integrals of X_Q.

The above preparations pave the way for the induction of relations in Section 16. The idea is as follows. Let f_t be the family defined by (0.2), and let a Laurent functional $\mathcal{L}_t$ on ${}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ be given for each $t \in W_Q$. Then by the vanishing theorem a relation of the form $\sum_t \mathcal{L}_t f_t = 0$ is valid if a similar relation is valid for the $(\lambda - \rho_Q)$-coefficients along Q; this in turn may be expressed as a similar relation between partial Eisenstein integrals for the lower split rank space X_Q. In this setting, taking the $(\lambda - \rho_Q)$-coefficient along Q essentially inverts the procedure of parabolic induction from Q to G. This motivaties our choice of terminology. The precise result is formulated in Theorem 16.1. An equivalent result, closer to the formulation of Casselman's theorem in [1] is stated at the end of the section.

1 Exponential polynomial series

Let A be a vectorial group and $\mathfrak{a}$ its Lie algebra. The exponential map $\exp\colon \mathfrak{a} \to A$ is a diffeomorphism; we denote its inverse by $\log$. If ξ belongs to $\mathfrak{a}_\mathbb{C}^*$, the complexified linear dual of $\mathfrak{a}$, then we define the function $e^\xi\colon a \mapsto a^\xi$ on A by $a^\xi = e^{\xi(\log a)}$. Let $P(\mathfrak{a})$ denote the algebra of polynomial functions $\mathfrak{a} \to \mathbb{C}$. If $d \in \mathbb{N}$, let $P_d(\mathfrak{a})$ denote the (finite dimensional) subspace of polynomials of degree at most d. Let Δ be a set of linearly independent vectors in $\mathfrak{a}$ (we do not require this set to span $\mathfrak{a}$). We put

$$\mathfrak{a}^+ = \mathfrak{a}^+(\Delta) := \{X \in \mathfrak{a} \mid \alpha(X) > 0, \quad \forall\, \alpha \in \Delta\},$$

and $A^+ = A^+(\Delta) = \exp(\mathfrak{a}^+)$. We define $\mathbb{N}\Delta$ to be the $\mathbb{N}$-span of Δ; if $\Delta = \emptyset$ then $\mathbb{N}\Delta = \{0\}$. Moreover, if X is a subset of $\mathfrak{a}_\mathbb{C}^*$, we denote by $X - \mathbb{N}\Delta$ the vectorial sum of X and $\mathbb{N}\Delta$.

Let V be a complete locally convex space; here and in the following we will always assume such a space to be Hausdorff. If $\xi \in \mathfrak{a}_\mathbb{C}^*$, then by a V-valued ξ-exponential polynomial function on A we mean a function $A \to V$ of the form $a \mapsto a^\xi q(\log a)$, with $q \in P(\mathfrak{a}) \otimes V$.

Definition 1.1 *By a Δ-exponential polynomial series on A with coefficients in V we mean a formal series F of exponential polynomial functions of the form*

$$\sum_{\xi \in \mathfrak{a}_\mathbb{C}^*} a^\xi\, q_\xi(\log a), \tag{1.1}$$

with $\xi \mapsto q_\xi$ a map $\mathfrak{a}_\mathbb{C}^* \to P(\mathfrak{a}) \otimes V$, such that

(a) *there exists a finite subset $X \subset \mathfrak{a}_\mathbb{C}^*$ such that $q_\xi = 0$ for $\xi \notin X - \mathbb{N}\Delta$;*

(b) *there exists a constant $d \in \mathbb{N}$ such that $q_\xi \in P_d(\mathfrak{a}) \otimes V$ for all $\xi \in \mathfrak{a}_\mathbb{C}^*$.*

The smallest $d \in \mathbb{N}$ with property (b) will be called the polynomial degree of the series; this number is denoted by $\deg(F)$.

The collection of all Δ-exponential polynomial series with coefficients in V is denoted by $\mathcal{F}^{\mathrm{ep}}(A, V) = \mathcal{F}^{\mathrm{ep}}_\Delta(A, V)$.

If $F \in \mathcal{F}^{\mathrm{ep}}(A, V)$ is an expansion of the form (1.1) then, for every $\xi \in \mathfrak{a}_\mathbb{C}^*$, we write $q_\xi(F)$ for q_ξ. Moreover, we write $q(F)$ for the map $\xi \mapsto q_\xi(F)$ from $\mathfrak{a}_\mathbb{C}^*$ to $P_d(\mathfrak{a}) \otimes V$. Then $F \mapsto q(F)$ defines a bijection from $\mathcal{F}^{\mathrm{ep}}(A, V)$ onto a linear subspace of $(P_d(\mathfrak{a}) \otimes V)^{\mathfrak{a}_\mathbb{C}^*}$, the space of maps $\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^* \to P_d(\mathfrak{a}) \otimes V$. Via this bijection we equip $\mathcal{F}^{\mathrm{ep}}(A, V)$ with the structure of a linear space.

If $F \in \mathcal{F}^{\mathrm{ep}}(A, V)$, then

$$\mathrm{Exp}(F) := \{\xi \in \mathfrak{a}_\mathbb{C}^* \mid q_\xi(F) \neq 0\}$$

is called the set of exponents of F. If $F_1, F_2 \in \mathcal{F}^{\mathrm{ep}}(A, V)$, we call F_1 a subseries of F_2 if $q_\xi(F_2) = q_\xi(F_1)$ for all $\xi \in \mathrm{Exp}(F_1)$.

The series (1.1) is said to converge absolutely in a fixed point $a_0 \in A$ if the series

$$\sum_{\xi \in \mathrm{Exp}(F)} a_0^\xi q_\xi(\log a_0)$$

with coefficients in V converges absolutely. It is said to converge absolutely on a subset $\Omega \subset A$ if it converges absolutely in every point $a_0 \in \Omega$. In this case pointwise summation of the series defines a function $\Omega \to V$.

We will also need a more special type of convergence for the series (1.1).

Definition 1.2 *The series (1.1) is said to converge neatly at a fixed point $a_0 \in A$ if for every continuous seminorm s on $P_d(\mathfrak{a}) \otimes V$, where $d = \deg(F)$, the series*

$$\sum_{\xi \in \mathrm{Exp}(F)} s(q_\xi) a_0^{\mathrm{Re}\,\xi}$$

converges.

The series (1.1) is said to converge neatly on a subset Ω of A if it converges neatly at every point of Ω.

Remark 1.3 If the series (1.1) converges neatly at a point $a_0 \in A$, then so does every subseries. Moreover, neat convergence at a_0 implies absolute convergence in a_0. However, we should warn the reader that neat convergence at a_0 cannot be seen from the series with coefficients in V arising from (1.1) by evaluation of its terms at $a = a_0$, since this type of convergence involves the global behavior of the polynomials q_ξ. In particular, it is possible that the series (1.1) does not converge neatly at a_0, whereas its evaluation in a_0 is identically zero.

The motivation for the definition of neat convergence is provided later by Lemmas 1.5 and 1.9, which express that neat convergence of the series (1.1) on an open subset $\Omega \subset A$ guarantees that (a) the function $f\colon \Omega \to V$ defined by (1.1) is real analytic on Ω; (b) its derivatives are given by series obtained by termwise differentiation from (1.1).

By a Δ-power series on A, with coefficients in V, we mean a Δ-exponential polynomial series F with $\deg F = 0$ and $\mathrm{Exp}(F) \subset -\mathbb{N}\Delta$, i.e.,

$$F = \sum_{\xi \in -\mathbb{N}\Delta} a^{\xi} c_{\xi}, \tag{1.2}$$

with $c_\xi \in V$, for $\xi \in -\mathbb{N}\Delta$. Note that for a Δ-power series the notion of neat convergence at a point $a_0 \in A$ coincides with the notion of absolute convergence in the point a_0.

The terminology 'power series' is motivated by the following consideration. If $\mu \in \mathbb{N}\Delta$, we put $\mu = \sum_{\alpha \in \Delta} \mu_\alpha \alpha$, with $\mu_\alpha \in \mathbb{N}$. For $z \in \mathbb{C}^\Delta$, we write

$$z^\mu = \prod_{\alpha \in \Delta} z_\alpha^{\mu_\alpha}.$$

Finally, to the series (1.2) we associate the power series

$$\sum_{\mu \in \mathbb{N}\Delta} z^\mu c_{-\mu} \tag{1.3}$$

with coefficients in V.

Let $\underline{z}\colon A \to \mathbb{C}^\Delta$ be the map defined by $\underline{z}(a)_\alpha = a^{-\alpha}$. Then it is obvious that the series (1.2) converges with sum S for $a = a_0$ if and only if the power series (1.3) converges with sum S for $z = \underline{z}(a_0)$. If $r \in]\,0, \infty\,[^\Delta$ we write $D(0, r)$ for the polydisc in $\mathbb{C}^\Delta$ consisting of the points z with $|z_\alpha| < r_\alpha$ for all $\alpha \in \Delta$. Note that the preimage of this set in A under the map $\underline{z}$ is given by

$$A^+(\Delta, r) := \{a \in A \mid a^{-\alpha} < r_\alpha, \ \ \forall\, \alpha \in \Delta\}.$$

If $R > 0$, we also agree to write $A^+(\Delta, R)$ for $A^+(\Delta, r)$ with r defined by $r_\alpha = R$ for all $\alpha \in \Delta$. Finally, if $a_0 \in A$, we write $A^+(\Delta, a_0) := A^+(\Delta, \underline{z}(a_0))$. Thus,

$$A^+(\Delta, a_0) := \{a \in A \mid a^\alpha > a_0^\alpha, \ \ \forall\, \alpha \in \Delta\} = A^+ a_0. \tag{1.4}$$

We now note that if (1.2) converges absolutely for $a = a_0$, then the power series (1.3) converges absolutely for $z = z(a_0)$, hence uniformly absolutely on the closure of the polydisc $D(0, \underline{z}(a_0))$. It follows that the series (1.2) then converges uniformly absolutely on the closure of $A^+(\Delta, a_0)$.

Let $a_0 \in A$. By $\mathcal{O}(A^+(\Delta, a_0), V)$ we denote the space of functions $f\colon A^+(\Delta, a_0) \to V$ that are given by an absolutely converging series of the form (1.2). For such a function the associated power series (1.3) converges absolutely on the polydisc

$D(0,r)$, with $r = \underline{z}(a_0)$; let $\tilde{f}\colon D(0,r) \to V$ be the holomorphic function defined by it. Then obviously

$$f(a) = \tilde{f}(\underline{z}(a)), \qquad (a \in A^+(\Delta, a_0)).$$

We see that the Δ-power series representing $f \in \mathcal{O}(A^+(\Delta, a_0)$ is unique. Moreover, let $\mathcal{O}(D(0,r), V)$ denote the space of holomorphic functions $D(0,r) \to V$, then it follows that the map

$$f \mapsto \tilde{f}, \quad \mathcal{O}(A^+(\Delta, r), V) \to \mathcal{O}(D(0,r), V)$$

is a linear isomorphism.

In particular it follows that every $f \in \mathcal{O}(A^+(\Delta, r), V)$ is real analytic on $A^+(\Delta, r)$. Moreover, its Δ-power series converges uniformly absolutely on every set of the form $A^+(\Delta, \rho)$, where $\rho \in]\,0, \infty\,[^\Delta$, $\rho_\alpha < r_\alpha$ for all $\alpha \in \Delta$.

If $\mathfrak{v}$ is a real linear space, then by $S(\mathfrak{v})$ we denote the symmetric algebra of its complexification $\mathfrak{v}_{\mathbb{C}}$. Via the right regular action we identify $S(\mathfrak{a})$ with the algebra of invariant differential operators on A. If $f \in \mathcal{O}(A^+(\Delta, r), V)$ and $u \in S(\mathfrak{a})$, then uf belongs to $\mathcal{O}(A^+(\Delta, r), V)$ again; its series may be obtained from the series of f by termwise application of u.

We now return to the more general exponential polynomial series (1.1) with coefficients in V. Let $d \geq \deg(F)$. Fix a basis Λ of $\mathfrak{a}$. For $m \in \mathbb{N}\Lambda$ we write $m = \sum_{\lambda \in \Lambda} m_\lambda \lambda$ and $|m| = \sum_\lambda m_\lambda$. For such m we define the polynomial function $X \mapsto X^m$ on $\mathfrak{a}$ by

$$X^m = \prod_{\lambda \in \Lambda} \lambda(X)^{m_\lambda}.$$

These polynomial functions with $|m| \leq d$ constitute a basis for $P_d(\mathfrak{a})$. Accordingly, we may write:

$$q_\xi(X) = \sum_{|m| \leq d} X^m c_{\xi,m}, \tag{1.5}$$

with $c_{\xi,m} \in V$.

Lemma 1.4 *The series (1.1) converges neatly on a set $\Omega \subset A$ if and only if for every $m \in \mathbb{N}\Lambda$ with $|m| \leq d$ the series*

$$\sum_{\xi \in \mathrm{Exp}(F)} a^\xi c_{\xi,m}$$

with coefficients in V converges absolutely for all $a \in \Omega$.

Proof: This is a straightforward consequence of the definition of neat convergence and the finite dimensionality of the space $P_d(\mathfrak{a})$. $\square$

We define a partial ordering $\preceq_\Delta$ on $\mathfrak{a}_\mathbb{C}^*$ by

$$\xi_1 \preceq_\Delta \xi_2 \iff \xi_2 - \xi_1 \in \mathbb{N}\Delta. \tag{1.6}$$

Moreover, we define the relation of Δ-integral equivalence on $\mathfrak{a}_\mathbb{C}^*$ by

$$\xi_1 \sim_\Delta \xi_2 \iff \xi_2 - \xi_1 \in \mathbb{Z}\Delta.$$

Let $F \in \mathcal{F}^{\mathrm{ep}}(A, V)$ be as in (1.1) and have polynomial degree at most d. In view of condition (a) of Definition 1.1, the restriction of the relation $\sim_\Delta$ to the set $\mathrm{Exp}(F)$ induces a finite partition of it. Every class ω in this partition has a least $\preceq_\Delta$-upper bound $s(\omega)$ in $\mathfrak{a}_\mathbb{C}^*$. Let $S = S_F$ be the set of these upper bounds. For every $s \in S$ and every $m \in \mathbb{N}\Lambda$ with $|m| \leq d$ we define the Δ-power series

$$f_{s,m}(a) = \sum_{\mu \in \mathbb{N}\Delta} a^{-\mu} c_{s-\mu,m}, \tag{1.7}$$

with coefficients determined by (1.5).

Lemma 1.5 *Let the series (1.1) be neatly convergent at the point $a_0 \in A$. Then the series (1.1) and, for every $s \in S = S_F$ and $m \in \mathbb{N}\Lambda$ with $|m| \leq d$, the series (1.7) is neatly convergent on the closure of the set $A^+(\Delta, a_0)$. The functions $f_{s,m}$, defined by (1.7), belong to $\mathcal{O}(A^+(\Delta, a_0), V)$. Moreover, let $f \colon A^+(\Delta, a_0) \to V$ be the function defined by the summation of (1.1). Then*

$$f(a) = \sum_{\substack{s \in S \\ |m| \leq d}} a^s (\log a)^m f_{s,m}(a), \qquad (a \in A^+(\Delta, a_0)). \tag{1.8}$$

In particular, the function $f \colon A^+(\Delta, a_0) \to V$ is real analytic.

Proof: From the neat convergence of (1.1) at a_0 it follows by Lemma 1.4 that for every s and m the series $\sum_{\mu \in \mathbb{N}\Delta} a^{s-\mu} c_{s-\mu,m}$ converges absolutely for $a = a_0$. This implies that the Δ-power series (1.7) converges absolutely for $a = a_0$. Hence it converges (uniformly) absolutely on the closure of $A^+(\Delta, a_0)$; in particular it converges neatly on that set. It follows from this that $f_{s,m} \in \mathcal{O}(A^+(\Delta, a_0), V)$, for $s \in S$ and $m \in \mathbb{N}\Lambda$ with $|m| \leq d$. Moreover,

$$a^s (\log a)^m f_{s,m} = \sum_{\xi \in s - \mathbb{N}\Delta} a^\xi (\log a)^m c_{\xi,m} \tag{1.9}$$

where the Δ-exponential polynomial series on the right-hand side converges neatly on the closure of $A^+(\Delta, a_0)$. The series (1.9), for $s \in S$ and $m \in \mathbb{N}\Lambda$ with $|m| \leq d$ add up to the series (1.1), which is therefore neatly convergent as well. Moreover, (1.8) follows. This in turn implies the real analyticity of the function f. $\square$

Remark 1.6 Let $\mathfrak{a}_\Delta := \cap_{\alpha \in \Delta} \ker \alpha$ and $A_\Delta := \exp(\mathfrak{a}_\Delta)$. Then the functions $f_{s,m}$, defined by (1.7) satisfy $f_{s,m}(aa_\Delta) = f_{s,m}(a)$ for all $a \in A$, $a_\Delta \in A_\Delta$. In particular, the function f of (1.8) generates a finite dimensional A_Δ-module with respect to the right regular action. Thus, if $\Delta = \emptyset$, then f is an exponential polynomial function.

Lemma 1.7 (Uniqueness of asymptotics) *Let $a_0 \in A$, and assume that the Δ-exponential polynomial series (1.1) converges neatly on $A^+(\Delta, a_0)$. If the sum of the series is zero for all $a \in A^+(\Delta, a_0)$, then $q_\xi = 0$ for all $\xi \in \mathfrak{a}_{\mathbb{C}}^*$.*

Proof: Let $f: A^+(\Delta, a_0) \to V$ be defined by summation of the series (1.1). Then it follows from Lemma 1.5 that the series (1.1) is an asymptotic expansion for f in the sense of [6], Sect. 3. Hence, if $f = 0$, then by uniqueness of asymptotics, see [22], p. 305, Cor. and [6], Prop. 3.1, it follows that the series vanishes identically. □

Definition 1.8 *Let $a_0 \in A$. By $C^{\mathrm{ep}}(A^+(\Delta, a_0), V)$ we denote the space of functions $f: A^+(\Delta, a_0) \to V$ that are given by the summation of a (necessarily unique) neatly converging Δ-exponential polynomial series of the form (1.1).*

If $f \in C^{\mathrm{ep}}(A^+(\Delta, a_0), V)$, then by $\mathrm{ep}(f)$ we denote the unique series from $\mathcal{F}^{\mathrm{ep}}(A, V)$ whose summation gives f. Moreover, the asymptotic degree of f is defined to be the number

$$\deg_{\mathrm{a}}(f) := \deg(\mathrm{ep}(f)).$$

Note that the map

$$\mathrm{ep}: C^{\mathrm{ep}}(A^+(\Delta, a_0), V) \to \mathcal{F}^{\mathrm{ep}}(A, V),$$

defined above, is a linear embedding.

Let $f \in C^{\mathrm{ep}}(A^+(\Delta, a_0), V)$. We briefly write $\mathrm{Exp}(f)$ for the set $\mathrm{Exp}(\mathrm{ep}(f))$; its elements are called the exponents of f. We put $q_\xi(f, \cdot) := q_\xi(\mathrm{ep}(f), \cdot)$, for $\xi \in \mathfrak{a}_{\mathbb{C}}^*$. Then $\xi \in \mathrm{Exp}(f) \iff q_\xi(f) \neq 0$.

The $\preceq_\Delta$-maximal elements in $\mathrm{Exp}(f)$ are called the (Δ-)leading exponents of f (or of the expansion). The set of these is denoted by $\mathrm{Exp}_{\mathrm{L}}(f)$.

By the formal application of $S(\mathfrak{a})$ to $\mathcal{F}^{\mathrm{ep}}(A, V)$ we shall mean the linear map

$$S(\mathfrak{a}) \otimes \mathcal{F}^{\mathrm{ep}}(A, V) \to \mathcal{F}^{\mathrm{ep}}(A, V)$$

induced by termwise differentiation (recall that $S(\mathfrak{a})$ acts on $C^\infty(A)$ via the right regular action). The image of an element $u \otimes F$ under this map will be denoted by uF.

Lemma 1.9 *Let $a_0 \in A$ and let $f \in C^{\mathrm{ep}}(A^+(\Delta, a_0), V)$. If $u \in S(\mathfrak{a})$ then the function $uf\colon a \mapsto R_u f(a)$ belongs to $C^{\mathrm{ep}}(A^+(\Delta, a_0), V)$. Moreover,*

$$\mathrm{ep}(uf) = u\,\mathrm{ep}(f).$$

Proof: We may assume that $u \in \mathfrak{a}$. Express f as in (1.8). For each s, m the function $uf_{s,m}$ belongs to $\mathcal{O}(A^+(\Delta, a_0), V)$; its expansion is obtained from $\mathrm{ep}(f_{s,m})$ by termwise application of u, hence by the formal application of u. $\square$

We shall also need a second type of formal application. Suppose that complete locally convex spaces U and W are given, and a continous bilinear map $U \times V \to W$, denoted by $(u, v) \mapsto uv$. By the formal application of $\mathcal{F}^{\mathrm{ep}}(A, U)$ to $\mathcal{F}^{\mathrm{ep}}(A, V)$ we mean the linear map

$$\mathcal{F}^{\mathrm{ep}}(A, U) \otimes \mathcal{F}^{\mathrm{ep}}(A, V) \to \mathcal{F}^{\mathrm{ep}}(A, W),$$

given by

$$\sum_{\xi \in \mathfrak{a}_{\mathbb{C}}^*} a^{\xi} p_{\xi}(\log a) \otimes \sum_{\eta \in \mathfrak{a}_{\mathrm{q}}^*} a^{\eta} q_{\eta}(\log a) \mapsto \sum_{\nu \in \mathfrak{a}_{\mathbb{C}}^*} a^{\nu} \sum_{\xi + \eta = \nu} p_{\xi}(\log a) q_{\eta}(\log a). \qquad (1.10)$$

This map is indeed well defined. To see this, let F denote the first series and G the second. Then for every $\nu \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, the collection S_ν of $(\xi, \eta) \in \mathrm{Exp}(F) \times \mathrm{Exp}(G)$ with $\xi + \eta = \nu$ is finite. Hence the W-valued polynomial function

$$r_\nu\colon X \mapsto \sum_{(\xi, \eta) \in S_\nu} p_\xi(X) q_\eta(X)$$

has degree at most $\deg(F) + \deg(G)$. Moreover, let $X_1, X_2 \subset \mathfrak{a}_{\mathbb{C}}^*$ be finite subsets such that $\mathrm{Exp}(F) \subset X_1 - \mathbb{N}\Delta$ and $\mathrm{Exp}(G) \subset X_2 - \mathbb{N}\Delta$ and put $X = X_1 + X_2$. Then for $\nu \in \mathfrak{a}_{\mathbb{C}}^* \setminus [X - \mathbb{N}\Delta]$ the collection S_ν is empty, hence $r_\nu = 0$. Therefore, the formal series on the right-hand side of (1.10) satisfies the conditions of Definition 1.1.

The image of an element $F \otimes G$ under the map (1.10) is denoted by FG. Again we have a lemma relating the formal application with neat convergence.

Lemma 1.10 *Let $U \times V \to W, (u, v) \mapsto uv$ be a continuous bilinear map of complete locally convex spaces. Let $a_0 \in A$ and let $f \in C^{\mathrm{ep}}(A(\Delta, a_0), U)$ and $g \in C^{\mathrm{ep}}(A(\Delta, a_0), V)$. Then the function $fg\colon a \mapsto f(a)g(a)$ belongs to $C^{\mathrm{ep}}(A(\Delta, a_0), W)$. Moreover, its Δ-exponential polynomial expansion is given by*

$$\mathrm{ep}(fg) = \mathrm{ep}(f)\,\mathrm{ep}(g).$$

Proof: This follows by a straightforward application of Lemma 1.5. $\square$

2 Basic notation, spherical functions

In this section we study spherical functions that are defined on a certain open dense subset X_+ of the symmetric space X, and are (radially) given by exponential polynomial series. This class of functions will play an important role in the paper. Later we will see that $\mathbb{D}(\mathrm{X})$-finite spherical funtions belong to this class.

Throughout this paper, we assume that X is a reductive symmetric space of Harish-Chandra's class, i.e., $\mathrm{X} = G/H$ with G a real reductive group of Harish-Chandra's class and H an open subgroup of G^σ, the group of fixed points for an involution σ of G. There exists a Cartan involution θ of G, commuting with σ. The associated fixed point group K is a σ-stable maximal compact subgroup.

We adopt the usual convention to denote Lie groups by Roman capitals and their Lie algebras by the corresponding Gothic lower cases. The infinitesimal involutions θ and σ of $\mathfrak{g}$ commute; let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q} \tag{2.1}$$

be the associated decompositions into $+1$ and -1 eigenspaces for θ and σ, respectively. We equip $\mathfrak{g}$ with a positive definite inner product $\langle \cdot\,,\,\cdot \rangle$ that is invariant under the compact group of automorphisms generated by $\mathrm{Ad}(K)$, $e^{i\mathrm{ad}\,(\mathfrak{p})}$, θ and σ. Then the decompositions (2.1) are orthogonal.

Let $\mathfrak{a}_\mathrm{q}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. We equip $\mathfrak{a}_\mathrm{q}$ with the restricted inner product $\langle \cdot\,,\,\cdot \rangle$ and its dual $\mathfrak{a}_\mathrm{q}^*$ with the dual inner product. The latter is extended to a complex bilinear form, also denoted $\langle \cdot\,,\,\cdot \rangle$, on the complexified dual $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

The exponential map is a diffeomorphism from $\mathfrak{a}_\mathrm{q}$ onto a vectorial subgroup A_q of G. We recall that $G = KA_\mathrm{q}H$. Let Σ be the restricted root system of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}$; we recall that the associated Weyl group W is naturally isomorphic to $N_K(\mathfrak{a}_\mathrm{q})/Z_K(\mathfrak{a}_\mathrm{q})$, the normalizer modulo the centralizer of $\mathfrak{a}_\mathrm{q}$ in K. Let $\mathfrak{a}_\mathrm{q}^{\mathrm{reg}}$ denote the associated set of regular elements in $\mathfrak{a}_\mathrm{q}$, i.e., the complement of the union of the root hyperplanes $\ker\alpha$, as $\alpha \in \Sigma$. We put $A_\mathrm{q}^{\mathrm{reg}} := \exp(\mathfrak{a}_\mathrm{q}^{\mathrm{reg}})$ and define the dense subset X_+ of X by

$$\mathrm{X}_+ = KA_\mathrm{q}^{\mathrm{reg}}H.$$

If Q is a parabolic subgroup of G, we denote its Langlands decomposition by $Q = M_QA_QN_Q$. By a σ-parabolic subgroup of G we mean a parabolic subgroup that is invariant under the composition $\sigma\circ\theta$. It follows from [4], Lemmas 2.5 and 2.6, that the collection $\mathcal{P}_\sigma$ of σ-parabolic subgroups of G containing A_q is finite.

If Q is a σ-parabolic subgroup then the Lie algebra $\mathfrak{a}_Q$ of its split component is σ-stable, hence decomposes as $\mathfrak{a}_Q = \mathfrak{a}_{Q\mathrm{h}} \oplus \mathfrak{a}_{Q\mathrm{q}}$, the vector sum of the associated $+1$ and -1 eigenspaces of $\sigma|_{\mathfrak{a}_Q}$, respectively. We write $A_{Q\mathrm{q}} := \exp\mathfrak{a}_{Q\mathrm{q}}$ and $M_{Q\sigma} := M_Q(A_Q\cap H)$; the decomposition

$$Q = M_{Q\sigma}A_{Q\mathrm{q}}N_Q$$

is called the σ-Langlands decomposition of Q. If $Q \in \mathcal{P}_\sigma$, then $M_{1Q} = Q \cap \theta(Q)$ contains $A_{\rm q}$. Hence $\mathfrak{a}_{Q{\rm q}}$ is contained in $\mathfrak{p} \cap \mathfrak{q}$ and centralizes $\mathfrak{a}_{\rm q}$; it follows that $\mathfrak{a}_{Q{\rm q}} \subset \mathfrak{a}_{\rm q}$. By $\Sigma(Q)$ we denote the set of roots of Σ occurring in $\mathfrak{n}_Q$. Obviously,

$$\mathfrak{n}_Q = \oplus_{\alpha \in \Sigma(Q)} \mathfrak{g}_\alpha.$$

Let $\mathcal{P}_\sigma^{\rm min}$ denote the collection of elements of $\mathcal{P}_\sigma$ that are minimal with respect to inclusion. An element $P \in \mathcal{P}_\sigma$ belongs to $\mathcal{P}_\sigma^{\rm min}$ if and only if $\mathfrak{a}_{P{\rm q}} = \mathfrak{a}_{\rm q}$, see [4], Cor. 2.7. This implies that the associated groups M_P and A_P are independent of $P \in \mathcal{P}_\sigma^{\rm min}$. We denote them by M and A, respectively. From the maximality of $\mathfrak{a}_{\rm q}$ in $\mathfrak{p} \cap \mathfrak{q}$ it follows that $\mathfrak{m} \cap \mathfrak{p} \subset \mathfrak{h}$. Thus, if $K_{\rm M} := K \cap M$ and $H_{\rm M} := H \cap M$, then the inclusion map $K_{\rm M} \to M$ induces a diffeomorphism

$$K_{\rm M}/K_{\rm M} \cap H \ \xrightarrow{\simeq}\ M/H_{\rm M}. \tag{2.2}$$

In particular, the symmetric space $M/H_{\rm M}$ is compact.

According to [4], Lemma 2.8, the map $P \mapsto \Sigma(P)$ induces a bijective map from $\mathcal{P}_\sigma^{\rm min}$ onto the collection of positive systems for Σ. If Φ is a positive system for Σ, then the associated element $P \in \mathcal{P}_\sigma^{\rm min}$ is given by the following characterization of its Lie algebra: ${\rm Lie}\,(P) = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$. From this we see that $N_K(\mathfrak{a}_{\rm q})$ acts on $\mathcal{P}_\sigma^{\rm min}$ by conjugation; moreover, the action commutes with the map $P \mapsto \Sigma(P)$. Accordingly, the action factors to a free transitive action of W on $\mathcal{P}_\sigma^{\rm min}$, see also [4], Lemma 2.8.

If $P \in \mathcal{P}_\sigma^{\rm min}$, then the collection of simple roots for the positive system $\Sigma(P)$ is denoted by $\Delta(P)$; the associated positive chamber in $\mathfrak{a}_{\rm q}$ is denoted by $\mathfrak{a}_{\rm q}^+(P)$ and the corresponding chamber in $A_{\rm q}$ by $A_{\rm q}^+(P)$. Thus, we see that $A_{\rm q}^{\rm reg}$ is the disjoint union of the chambers $A_{\rm q}^+(P)$, as $P \in \mathcal{P}_\sigma^{\rm min}$.

More generally, if $Q \in \mathcal{P}_\sigma$, we write

$$\mathfrak{a}_{Q{\rm q}}^+ := \{X \in \mathfrak{a}_{Q{\rm q}} \mid \alpha(X) > 0 \quad \text{for} \quad \alpha \in \Sigma(Q)\}. \tag{2.3}$$

It follows from [4], Lemmas 2.5 and 2.6, that $\mathfrak{a}_{Q{\rm q}}^+ \neq \emptyset$. Moreover, if $X \in \mathfrak{a}_{Q{\rm q}}^+$, then the parabolic subgroup Q is determined by the following characterization of its Lie algebra

$${\rm Lie}\,(Q) = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\substack{\alpha \in \Sigma \\ \alpha(X) \geq 0}} \mathfrak{g}_\alpha. \tag{2.4}$$

Conversely, if X is any element of $\mathfrak{a}_{\rm q}$, then (2.4) defines the Lie algebra of a group Q from $\mathcal{P}_\sigma$; moreover, $X \in \mathfrak{a}_{Q{\rm q}}^+$. From this we readily see that conjugation induces an action of $N_K(\mathfrak{a}_{\rm q})$ on $\mathcal{P}_\sigma$, which factors to an action of W.

By a straightforward calculation involving root spaces, it follows that the multiplication map $K \times A_{\rm q}^{\rm reg} \to {\rm X}$ induces a diffeomorphism

$$K \times_{N_K(\mathfrak{a}_{\rm q}) \cap H} A_{\rm q}^{\rm reg} \ \xrightarrow{\simeq}\ {\rm X}_+.$$

In particular, it follows that X_+ is an open dense subset of X. Let $W_{K\cap H}$ denote the canonical image of $N_K(\mathfrak{a}_\mathrm{q}) \cap H$ in W and let $\mathcal{W}$ be a complete set of representatives for $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_\mathrm{q})$. If $P \in \mathcal{P}_\sigma^{\min}$, then it follows that

$$\mathrm{X}_+ = \cup_{w\in\mathcal{W}} \;\; KA_\mathrm{q}^+(P)wH \qquad \text{(disjoint union)}. \tag{2.5}$$

Moreover, for each $w \in \mathcal{W}$ the multiplication map $(k, a) \mapsto kawH$ induces a diffeomorphism

$$K \times_{K_\mathrm{M}\cap wHw^{-1}} A_\mathrm{q}^+(P) \xrightarrow{\simeq} KA_\mathrm{q}^+(P)wH. \tag{2.6}$$

Here we have written $K_\mathrm{M} = K \cap M$; in (2.6) the set on the right is an open subset of X.

Let (τ, V_τ) be a smooth representation of K in a complete locally convex space. For later applications it will be crucial that we allow τ to be infinite dimensional (see the proof of Theorem 7.7).

By $C^\infty(\mathrm{X}_+ : \tau)$ we denote the space of smooth functions $f\colon \mathrm{X}_+ \to V_\tau$ that are τ-spherical, i.e.,

$$f(kx) = \tau(k)f(x), \tag{2.7}$$

for $x \in \mathrm{X}_+$, $k \in K$. The space $C^\infty(\mathrm{X} : \tau)$ of smooth τ-spherical functions on X will be identified with the subspace of functions in $C^\infty(\mathrm{X}_+ : \tau)$ that extend smoothly to all of X.

In the following we assume that $P \in \mathcal{P}_\sigma^{\min}$ is fixed. If $w \in N_K(\mathfrak{a}_\mathrm{q})$, then by $C^\infty_{P,w}(\mathrm{X}_+ : \tau)$ or $C^\infty_w(\mathrm{X}_+ : \tau)$ we denote the space of functions $f \in C^\infty(\mathrm{X}_+ : \tau)$ with support contained in $KA_\mathrm{q}^+(P)wH$. From (2.5) we see that

$$C^\infty(\mathrm{X}_+ : \tau) = \oplus_{w\in\mathcal{W}} \;\; C^\infty_w(\mathrm{X}_+ : \tau).$$

Let $w \in N_K(\mathfrak{a}_\mathrm{q})$ be fixed for the moment. For $f \in C^\infty(\mathrm{X}_+ : \tau)$ we define the function $T^\downarrow_{P,w} f \in C^\infty(A_\mathrm{q}^+(P), V_\tau^{K_\mathrm{M}\cap wHw^{-1}})$ by

$$T^\downarrow_{P,w} f(a) = f(awH).$$

Since (2.6) is a diffeomorphism, the restriction of $T^\downarrow_{P,w}$ to $C^\infty_w(\mathrm{X}_+ : \tau)$ is an isomorphism of complete locally convex spaces onto the space $C^\infty(A_\mathrm{q}^+(P), V_\tau^{K_\mathrm{M}\cap wHw^{-1}})$. Taking the direct sum of the maps $T^\downarrow_{P,w}$, as $w \in \mathcal{W}$, we therefore obtain an isomorphism of complete locally convex spaces

$$T^\downarrow_{P,\mathcal{W}}\colon \; C^\infty(\mathrm{X}_+ : \tau) \xrightarrow{\simeq} \oplus_{w\in\mathcal{W}} \;\; C^\infty(A_\mathrm{q}^+(P), V_\tau^{K_\mathrm{M}\cap wHw^{-1}}). \tag{2.8}$$

Definition 2.1 *We denote by $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ the space of functions $f \in C^\infty(\mathrm{X}_+ : \tau)$ such that for every $w \in \mathcal{W}$ the function $T^\downarrow_{P,w} f$ belongs to $C^{\mathrm{ep}}(A_\mathrm{q}^+(P), V_\tau^{K_\mathrm{M}\cap wHw^{-1}})$, where the latter space is defined as in Definition 1.8, with $\mathfrak{a}$, a_0 and Δ replaced by $\mathfrak{a}_\mathrm{q}$, e and $\Delta(P)$, respectively.*

If $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, we define its asymptotic degree to be the number

$$\deg_{\mathrm{a}}(f) := \max_{w \in \mathcal{W}} \deg(T^{\downarrow}_{P,w} f).$$

It follows from the above definition that restriction of $T^{\downarrow}_{P,\mathcal{W}}$ induces a linear isomorphism

$$C^{\mathrm{ep}}(\mathrm{X}_+ : \tau) \simeq \oplus_{w \in \mathcal{W}} \; C^{\mathrm{ep}}(A^+_{\mathrm{q}}(P), V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}). \tag{2.9}$$

Using conjugations by elements of $N_K(\mathfrak{a}_{\mathrm{q}})$ it is readily seen that the space $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ and the map $\deg_{\mathrm{a}} : C^{\mathrm{ep}}(\mathrm{X}_+ : \tau) \to \mathbb{N}$ are independent of the particular choices of P and $\mathcal{W}$. In particular, if $P \in \mathcal{P}_\sigma^{\min}$ and $w \in N_K(\mathfrak{a}_{\mathrm{q}})$, then $T^{\downarrow}_{P,w} f \in C^{\mathrm{ep}}(A^+_{\mathrm{q}}(P), V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}})$ and $\deg(T^{\downarrow}_{P,w} f) \leq \deg_{\mathrm{a}}(f)$. We put

$$\mathrm{Exp}(P, w \,|\, f) := \mathrm{Exp}(T^{\downarrow}_{P,w} f), \qquad \text{and} \qquad \mathrm{Exp}_{\mathrm{L}}(P, w \,|\, f) := \mathrm{Exp}_{\mathrm{L}}(T^{\downarrow}_{P,w} f).$$

Moreover, for all $\xi \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ we define $\underline{q}_\xi(P, w \,|\, f) = q_\xi(T^{\downarrow}_{P,w} f)$. Then, for every $a \in A^+_{\mathrm{q}}(P)$,

$$f(aw) = \sum_{\xi \in \mathrm{Exp}(P,w \,|\, f)} a^\xi \, \underline{q}_\xi(P, w \,|\, f, \log a), \tag{2.10}$$

where the $\Delta(P)$-exponential polynomial series on the right-hand side neatly converges on $A^+_{\mathrm{q}}(P)$.

For $w \in N_K(\mathfrak{a}_{\mathrm{q}})$, we will use the notation

$$\mathrm{X}_{0,w} := M / M \cap wHw^{-1}; \tag{2.11}$$

moreover, we put $\tau_{\mathrm{M}} := \tau_{K_{\mathrm{M}}}$ and write $C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}})$ for the space of τ_{M}-spherical C^∞ functions from $\mathrm{X}_{0,w}$ to V_τ, i.e., the space of functions $f \in C^\infty(\mathrm{X}_{0,w}, V_\tau)$ satisfying the rule (2.7) for $k \in K_{\mathrm{M}}$ and $x \in \mathrm{X}_{0,w}$. From (2.2) with wHw^{-1} in place of H we see that the inclusion $K_{\mathrm{M}} \to M$ induces a diffemorphism from $K_{\mathrm{M}}/K_{\mathrm{M}} \cap wHw^{-1}$ onto $\mathrm{X}_{0,w}$. Hence evaluation at the point $e(M \cap wHw^{-1})$ induces a linear isomorphism from $C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}})$ onto $V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}$. Thus, if $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, then for every $\xi \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}$ there exists a unique $C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}})$-valued polynomial function $q_\xi(P, w \,|\, f)$ on $\mathfrak{a}_{\mathrm{q}}$ such that

$$q_\xi(P, w \,|\, f, X, e) = \underline{q}_\xi(P, w \,|\, f)(X) \qquad (X \in \mathfrak{a}_{\mathrm{q}}).$$

Using sphericality of the function f we obtain from (2.10) that

$$f(maw) = \sum_{\xi \in \mathrm{Exp}(P,w \,|\, f)} a^\xi q_\xi(P, w \,|\, f, \log a, m), \tag{2.12}$$

for $m \in M, \; a \in A^+_{\mathrm{q}}(P)$. The series on the right-hand side is a $\Delta(P)$-exponential polynomial series in the variable a, with coefficients in $C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}})$, relative to the variable m. As such it converges neatly on $A^+_{\mathrm{q}}(P)$.

We shall now discuss a lemma whose main purpose is to enable us to reduce on the set of exponents in certain proofs, in order to simplify the exposition.

Lemma 2.2 *Let $P \in \mathcal{P}_\sigma^{\min}$ and let $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ be a complete set of representatives of $W/W_{K\cap H}$. Assume that $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$.*

There exists a finite set $S \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ of mutually $\Delta(P)$-integrally inequivalent elements such that $\mathrm{Exp}(P, v \,|\, f) \subset S - \mathbb{N}\Delta(P)$ for every $v \in \mathcal{W}$.

If S is a set as above, then there exist unique functions $f_s \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, for $s \in S$, such that

$$f = \sum_{s\in S} f_s,$$

and such that $\mathrm{Exp}(P, v \,|\, f_s) \subset s - \mathbb{N}\Delta(P)$, for every $v \in \mathcal{W}$.

Proof: There exists a finite set $X \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ such that $\mathrm{Exp}(P, v \,|\, f) \subset X - \mathbb{N}\Delta(P)$ for all $v \in \mathcal{W}$. Obviously there exists a finite set S as required, such that $X - \mathbb{N}\Delta(P) \subset S - \mathbb{N}\Delta(P)$.

If S is such as mentioned, then for $s \in S$ and $v \in \mathcal{W}$ we define the function $f_{s,v}\colon A_{\mathrm{q}}^+(P) \to V_\tau^{K_{\mathrm{M}} \cap vHv^{-1}}$ by

$$f_{s,v}(a) = \sum_{\nu\in\mathbb{N}\Delta(P)} a^{s-\nu} q_{s-\nu}(P, v \mid f, \log a, e);$$

here the exponential polynomial series is neatly convergent, hence $f_{s,v}$ belongs to the space $C^{\mathrm{ep}}(A_{\mathrm{q}}^+(P), V_\tau^{K_{\mathrm{M}}\cap vHv^{-1}})$, for every $v \in \mathcal{W}$. By the isomorphism (2.9) there exists a unique function $f_s \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ such that $f_s(av) = f_{s,v}(a)$ for $v \in \mathcal{W}$, $a \in A_{\mathrm{q}}^+(P)$. By the hypothesis on S the sets $s - \mathbb{N}\Delta(P)$, for $s \in S$, are disjoint. Hence $f = \sum_{s\in S} f_s$ on $A_{\mathrm{q}}^+(P)v$, for every $v \in \mathcal{W}$. By (2.5) and sphericality this equality holds on all of X_+. $\square$

3 Asymptotic behavior along walls

In this section we study the asymptotic behavior along walls of functions from $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$; here τ is a smooth representation in a complete locally convex space V_τ.

Let $P \in \mathcal{P}_\sigma^{\min}$ and let Q be a σ-parabolic subgroup with Langlands decomposition $Q = M_Q A_Q N_Q$, containing P. In addition to the notation introduced in the beginning of the previous section, the following notation will also be convenient.

We agree to write $K_Q{:}= K \cap M_Q$ and $H_Q{:}= H \cap M_Q$. Moreover, W_Q denotes the centralizer of $\mathfrak{a}_{Q\mathrm{q}}$ in W. Then $W_Q \simeq N_{K_Q}(\mathfrak{a}_{\mathrm{q}})/Z_{K_Q}(\mathfrak{a}_{\mathrm{q}})$. On the other hand W_Q is also the subgroup of W generated by the reflections in the roots from the set

$$\Delta_Q(P){:}= \{\alpha \in \Delta(P) \mid \alpha|_{\mathfrak{a}_{Q\mathrm{q}}} = 0\}.$$

We note that $\Sigma(Q) = \Sigma(P)\backslash\mathbb{N}\Delta_Q(P)$. Moreover, let $\Sigma_r(Q)$ denote the collection of $\mathfrak{a}_{Q\mathrm{q}}$-weights in $\mathfrak{n}_Q$. Then

$$\Sigma_r(Q) = \{\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \mid \alpha \in \Sigma(Q)\}.$$

Let $\Delta_r(Q)$ be the collection of weights from the set $\Sigma_r(Q)$ that cannot be written as the sum of two weights from that set; then one readily verifies that $\Delta_r(Q)$ equals the set of restrictions of elements from $\Delta(P) \setminus \Delta_Q(P)$ to $\mathfrak{a}_{Q\mathrm{q}}$. In particular, the elements of $\Delta_r(Q)$ are linearly independent.

Given $a_0 \in A_{Q\mathrm{q}}$ we shall briefly write $A^+_{Q\mathrm{q}}(a_0)$ for the set $A^+_{Q\mathrm{q}}(\Delta_r(Q), a_0)$ defined as in (1.4) with $\mathfrak{a}_{Q\mathrm{q}}$ and $\Delta_r(Q)$ in place of $\mathfrak{a}$ and Δ, respectively. Similarly, if $\rho \in \,]\,0, \infty\,[^{\Delta_r(Q)}$, we briefly write

$$A^+_{Q\mathrm{q}}(\rho) := A^+_{Q\mathrm{q}}(\Delta_r(Q), \rho) = \{a \in A_{Q\mathrm{q}} \mid a^{-\alpha} < \rho_\alpha, \;\; \forall \alpha \in \Delta_r(Q)\}.$$

If $R > 0$, we write $A^+_{Q\mathrm{q}}(R)$ for $A^+_{Q\mathrm{q}}(\rho)$, where ρ is defined by $\rho_\alpha = R$ for every $\alpha \in \Delta_r(Q)$. Note that $A^+_{Q\mathrm{q}}(1)$ equals the positive chamber $A^+_{Q\mathrm{q}} := \exp(\mathfrak{a}^+_{Q\mathrm{q}})$, see (2.3).

If $v \in N_K(\mathfrak{a}_\mathrm{q})$, we define

$$\mathrm{X}_{1Q,v} := M_{1Q}/M_{1Q} \cap vHv^{-1}. \tag{3.1}$$

This is a symmetric space for the involution σ^v of M_{1Q} defined by $\sigma^v(m) = v\sigma(v^{-1}mv)v^{-1}$. Note that this involution commutes with the Cartan involution $\theta|_{M_{1Q}}$. Note also that $\mathfrak{a}_\mathrm{q}$ is a maximal abelian subspace of $\mathrm{Ad}(v)(\mathfrak{p} \cap \mathfrak{q}) = \mathfrak{p} \cap \mathrm{Ad}(v)\mathfrak{q}$. Hence it is the analogue of $\mathfrak{a}_\mathrm{q}$ for the triple $(M_{1Q}, K_Q, M_{1Q} \cap vHv^{-1})$. The corresponding group A_q may naturally be identified with a subspace of $\mathrm{X}_{1Q,v}$.

The image of M_Q in $\mathrm{X}_{1Q,v}$ may be identified with

$$\mathrm{X}_{Q,v} := M_Q/M_Q \cap vHv^{-1},$$

the symmetric space for the involution $\sigma^v|_{M_Q}$. It follows from the characterization of $\mathcal{P}_\sigma$ expressed by (2.4) that

$$\mathcal{P}_\sigma = \mathcal{P}_{\sigma^v} \tag{3.2}$$

Hence Q is a σ^v-parabolic subgroup as well. Hence $\mathfrak{a}_Q \cap \mathrm{Ad}(v)\mathfrak{q} = \mathfrak{a}_Q \cap \mathfrak{a}_\mathrm{q} = \mathfrak{a}_{Q\mathrm{q}}$, and we deduce that the inclusion $A_{Q\mathrm{q}} \to A_Q$ induces a diffeomorphism $A_{Q\mathrm{q}} \simeq A_Q/A_Q \cap vHv^{-1}$. From this we conclude that the multiplication map $M_Q \times A_{Q\mathrm{q}} \to M_{1Q}$ induces the decomposition

$$\mathrm{X}_{1Q,v} \simeq \mathrm{X}_{Q,v} \times A_{Q\mathrm{q}}. \tag{3.3}$$

Remark 3.1 In particular, the above definitions cover the two extreme cases that Q is minimal and that it equals G.

In the case that $Q \in \mathcal{P}^{\min}_\sigma$, we have $Q = MAN_Q$, and $\mathrm{X}_{Q,v}$ equals the space $\mathrm{X}_{0,v}$ defined in (2.11). Moreover, $\mathrm{X}_{1Q,v} \simeq \mathrm{X}_{0,v} \times A_\mathrm{q}$.

In the other extreme case we have $\mathrm{X}_{1G,v} = G/vHv^{-1}$. This symmetric space will also be denoted by X_v. Note that right multiplication by v induces an isomorphism of X_v onto X. Note also that M_G equals ${}^\circ G$, the intersection of

$\ker\chi$, as χ ranges over the positive characters of G. Hence $\mathrm{X}_{G,v} = {}^\circ G/{}^\circ G \cap vHv^{-1}$. Finally, $\mathrm{X}_v \simeq \mathrm{X}_{G,v} \times A_{G\mathrm{q}}$, where $A_{G\mathrm{q}}$ is the image under exp of the space $\mathfrak{a}_{G\mathrm{q}}$, which in turn is the intersection of the root hyperplanes $\ker\alpha$ as $\alpha \in \Sigma$.

Let $\bar{\mathfrak{n}}_Q := \theta\mathfrak{n}_Q$ be equipped with the restriction of the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ from $\mathfrak{g}$. If $Q \neq G$ we define the function $R_{Q,v}\colon M_{1Q} \to]0,\infty[$ by

$$R_{Q,v}(m) = \|\mathrm{Ad}(m\sigma^v(m)^{-1})|_{\bar{\mathfrak{n}}_Q}\|_{\mathrm{op}}^{1/2},$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm. We define $R_{G,v}$ to be the constant function 1.

The function $R_{Q,v}$ is right $M_{1Q} \cap vHv^{-1}$-invariant. It may therefore also be viewed as a function on $\mathrm{X}_{1Q,v}$. We shall describe the function $R_{Q,v}$ in more detail below.

The orthocomplement of $\mathfrak{a}_{Q\mathrm{q}}$ in $\mathfrak{a}_\mathrm{q}$ is denoted by ${}^*\mathfrak{a}_{Q\mathrm{q}}$. Note that

$${}^*\mathfrak{a}_{Q\mathrm{q}} = \mathfrak{m}_Q \cap \mathfrak{a}_\mathrm{q}; \tag{3.4}$$

hence ${}^*\mathfrak{a}_{Q\mathrm{q}}$ is the analogue of $\mathfrak{a}_\mathrm{q}$ for the triple (M_Q, K_Q, H_Q). We recall from the text following (3.1) that $\mathfrak{a}_\mathrm{q}$ is maximal abelian in $\mathfrak{p}\cap\mathrm{Ad}(v)\mathfrak{q}$ hence is the analogue of $\mathfrak{a}_\mathrm{q}$ for the triple (G, K, vHv^{-1}). Accordingly, ${}^*\mathfrak{a}_{Q\mathrm{q}}$ is also the analogue of $\mathfrak{a}_\mathrm{q}$ for the triple $(M_Q, K_Q, M_Q \cap vHv^{-1})$.

In view of (3.2), the group ${}^*P = P \cap M_Q$ is readily seen to be a minimal σ^v-parabolic subgroup for M_Q; the associated positive chamber in ${}^*A_{Q\mathrm{q}} = \exp({}^*\mathfrak{a}_{Q\mathrm{q}})$ is denoted by ${}^*A^+_{Q\mathrm{q}}({}^*P)$.

Let $\mathcal{W}_{Q,v}$ be an analogue for $\mathrm{X}_{Q,v}$ of $\mathcal{W}$, that is, $\mathcal{W}_{Q,v}$ is a complete set of representatives in $N_{K_Q}(\mathfrak{a}_\mathrm{q})$ for the quotient $W_Q/W_{K_Q\cap vHv^{-1}}$. Let $\mathrm{X}_{Q,v,+}$ be the analogue for $\mathrm{X}_{Q,v}$ of the open dense subset X_+ of X. According to (2.5) this set may be expressed as the following disjoint union of open subsets of $\mathrm{X}_{Q,v}$

$$\mathrm{X}_{Q,v,+} := \bigcup_{u\in\mathcal{W}_{Q,v}} K_Q{}^*A^+_{Q\mathrm{q}}({}^*P)\,u\,(M_Q \cap vHv^{-1}) \qquad \text{(disjoint union).} \tag{3.5}$$

Let $\mathrm{X}_{1Q,v,+}$ be the analogue of X_+ for $\mathrm{X}_{1Q,v}$; then from (3.3) we see that $\mathrm{X}_{1Q,v,+} \simeq \mathrm{X}_{Q,v,+} \times A_{Q\mathrm{q}}$. In terms of this decomposition and (3.5) the function $R_{Q,v}$ may be expressed as follows.

Lemma 3.2 *The function $R_{Q,v}\colon M_{1Q} \to]0,\infty[$ is continuous, and right $M_{1Q} \cap vHv^{-1}$- and left K_Q-invariant. Moreover, if $Q \neq G$ and if $a \in A_\mathrm{q}$ and $u \in N_{K_Q}(\mathfrak{a}_\mathrm{q})$, then*

$$R_{Q,v}(au) = \max_{\alpha\in\Sigma(Q)} a^{-\alpha}. \tag{3.6}$$

Finally, $R_{Q,v} \geq 1$ on $\mathrm{X}_{Q,v}$.

Proof: Since $R_{G,v}$ is the constant function 1, we may as well assume that $Q \neq G$. Continuity of the function $R_{Q,v}$ is obvious from its definition. The group K_Q is σ^v invariant and acts unitarily on $\bar{\mathfrak{n}}_Q$; hence the left K_Q-invariance is obvious from the definition. If $a \in A_{\mathrm{q}}$, then $a\sigma^v(a)^{-1} = a^2$. Hence the operator norm of $\mathrm{Ad}(a\sigma^v(a)^{-1})$ on $\bar{\mathfrak{n}}_Q$ equals the maximal value of $a^{-2\alpha}$ as $\alpha \in \Sigma(Q)$. This implies (3.6) for $u = 1$.

The element $u \in N_{K_Q}(\mathfrak{a}_{\mathrm{q}})$ belongs to M_Q, hence $\mathrm{Ad}(u)$ normalizes $\mathfrak{n}_Q$. Therefore, $\mathrm{Ad}(u)$ leaves the collection $\Sigma(Q)$ of $\mathfrak{a}_{\mathrm{q}}$-roots in $\mathfrak{n}_Q$ invariant. Put $a' = u^{-1}au$. Then $R_{Q,v}(au) = R_{Q,v}(a') = \max_{\alpha\in\Sigma(Q)}(a')^{-\alpha}$. Since $\mathrm{Ad}(u)$ leaves $\Sigma(Q)$ invariant, (3.6) follows.

If $\alpha \in \Sigma$, let h_α be the element of $\mathfrak{a}_{\mathrm{q}}$ determined by $\alpha(X) = \langle h_\alpha\,,\, X\rangle$, for $X \in \mathfrak{a}_{\mathrm{q}}$. Then the closure of ${}^*\mathfrak{a}^+_{Q\mathrm{q}}({}^*P)$ is contained in the closed convex cone generated by the elements h_β, for $\beta \in \Delta_Q(P)$. If $\alpha \in \Delta(P) \setminus \Delta_Q(P)$, then $\alpha(h_\beta) = \langle\alpha\,,\,\beta\rangle \leq 0$, for $\beta \in \Delta_Q(P)$; hence $\alpha \leq 0$ on ${}^*\mathfrak{a}^+_{Q\mathrm{q}}({}^*P)$. But $\Delta(P) \setminus \Delta_Q(P) \subset \Sigma(Q)$, hence it follows that $R_{Q,v} \geq 1$ on ${}^*A^+_{Q\mathrm{q}}({}^*P)u$, for every $u \in \mathcal{W}_{Q,v}$. The final assertion follows from combining this observation with (3.5), the left K_Q-invariance of $R_{Q,v}$ and density of $\mathrm{X}_{Q,v,+}$ in $\mathrm{X}_{Q,v}$. $\square$

If $1 \leq R \leq \infty$ we define

$$\mathrm{X}_{Q,v}[R] := \{m \in \mathrm{X}_{Q,v} \mid R_{Q,v}(m) < R\}. \tag{3.7}$$

Note that $\mathrm{X}_{Q,v}[1] = \emptyset$ and $\mathrm{X}_{Q,v}[\infty] = \mathrm{X}_{Q,v}$; moreover, $R_1 < R_2 \Rightarrow \mathrm{X}_{Q,v}[R_1] \subset \mathrm{X}_{Q,v}[R_2]$. Finally, the union of the sets $\mathrm{X}_{Q,v}[R]$ as $1 \leq R < \infty$ equals $\mathrm{X}_{Q,v}$.

In accordance with (3.7) we define $\mathrm{X}_{Q,v,+}[R] := \mathrm{X}_{Q,v,+} \cap \mathrm{X}_{Q,v}[R]$, for $1 \leq R \leq \infty$. Moreover, we also put

$${}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]} := \{a \in {}^*A^+_{Q\mathrm{q}}({}^*P) \mid a^{-\alpha} < R,\ \ \forall\alpha \in \Sigma(Q)\}.$$

Note that, if $\alpha \in \Sigma(P) \setminus \Sigma(Q)$, then $a^{-\alpha} < 1 \leq R$ for all $a \in {}^*A^+_{Q\mathrm{q}}({}^*P)$. Hence

$${}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]} = {}^*A^+_{Q\mathrm{q}}({}^*P) \cap A^+_{\mathrm{q}}(P, R).$$

It follows from (3.5) and Lemma 3.2 that

$$\mathrm{X}_{Q,v,+}[R] = \bigcup_{u\in\mathcal{W}_{Q,v}} K_Q\, {}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]}\, u\, (M_Q \cap vHv^{-1}) \qquad \text{(disjoint union).} \tag{3.8}$$

The function $R_{Q,v}$ plays a role in the description of the asymptotic behavior of a function $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ along 'the wall' $A^+_{Q\mathrm{q}}v$. This behaviour is described in terms of an expansion of $f(mav)$ in the variable $a \in A^+_{Q\mathrm{q}}$, for $m \in \mathrm{X}_{Q,v,+}$. Thus, it is of interest to know when $mavH$ belongs to X_+, the domain of f.

Lemma 3.3

(a) *If $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(b)^{-1})$ then $ba \in A^+_{\mathrm{q}}(P)$.*

(b) *Let* $m \in \mathrm{X}_{Q,v,+}$*. Then* $mavH \in \mathrm{X}_+$ *for all* $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$.

(c) *Let* $R \geq 1$*. Then* $\mathrm{X}_{Q,v,+}[R]A^+_{Q\mathrm{q}}(R^{-1})vH \subset \mathrm{X}_+$.

Proof: Let b and a fulfill the hypotheses of (a). If $\alpha \in \Delta_Q(P)$, then $(ba)^{-\alpha} = b^{-\alpha} < 1$. On the other hand we have, for $\alpha \in \Delta(P) \setminus \Delta_Q(P)$, that $\alpha \in \Sigma(Q)$, hence $(ba)^{-\alpha} \leq R_{Q,v}(b)a^{-\alpha} < 1$, by Lemma 3.2. Hence $ba \in A^+_{\mathrm{q}}(P)$, and (a) is proved.

Let m be as in (b), and let $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$. In view of (3.5) we may write $m = kbuh$ with $k \in K_Q$, $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$, $u \in \mathcal{W}_{Q,v}$ and $h \in M_Q \cap vHv^{-1}$. Now $mavH = kbuhavH = kbauvH$. Thus, it suffices to show that $ba \in A^+_{\mathrm{q}}(P)$. This follows from (a) and the observation that $R_{Q,v}(b) = R_{Q,v}(m)$, by Lemma 3.2.

Finally, (c) is a straightforward consequence of (b). $\square$

If $Q \in \mathcal{P}_\sigma$ we put $\tau_Q := \tau|_{K_Q}$. Then, for $v \in N_K(\mathfrak{a}_\mathrm{q})$, the space $C^{\mathrm{ep}}(\mathrm{X}_{Q,v,+} : \tau_Q)$ is defined as above (2.9) with $\mathrm{X}_{Q,v}$ and τ_Q in place of X and τ, respectively.

Theorem 3.4 *Let* $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$*. Let* $Q \in \mathcal{P}_\sigma$ *and* $v \in N_K(\mathfrak{a}_\mathrm{q})$.

(a) *There exist a constant* $k \in \mathbb{N}$*, a finite set* $Y \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$*, and for each* $\eta \in Y - \mathbb{N}\Delta_r(Q)$ *a* $C(\mathrm{X}_{Q,v,+}, V_\tau)$*-valued polynomial function* $q_\eta = q_\eta(Q, v \,|\, f)$ *on* $\mathfrak{a}_{Q\mathrm{q}}$ *of degree at most* k*, such that for every* $m \in \mathrm{X}_{Q,v,+}$

$$f(mav) = \sum_{\eta \in Y - \mathbb{N}\Delta_r(Q)} a^\eta q_\eta(\log a, m), \qquad (a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})), \quad (3.9)$$

where the $\Delta_r(Q)$*-exponential polynomial series with coefficients in* V_τ *converges neatly on the indicated subset of* $A_{Q\mathrm{q}}$.

(b) *The set* $\mathrm{Exp}(Q, v \,|\, f) := \{\eta \in Y - \mathbb{N}\Delta_r(Q) \mid q_\eta \neq 0\}$ *is uniquely determined. Moreover, the functions* q_η*, where* $\eta \in Y - \mathbb{N}\Delta_r(Q)$*, are unique and belong to* $P_d(\mathfrak{a}_{Q\mathrm{q}}) \otimes C^{\mathrm{ep}}(\mathrm{X}_{Q,v,+} : \tau_Q)$*, where* $d := \deg_\mathrm{a}(f)$*. Finally, if* $R > 1$*, then the the series on the right-hand side of (3.9) converges neatly on* $A^+_{Q\mathrm{q}}(R^{-1})$ *as a* $\Delta_r(Q)$*-exponential polynomial series with coefficients in* $C^\infty(\mathrm{X}_{Q,v,+}[R] : \tau_Q)$.

Proof: We will establish existence. Uniqueness then follows from uniqueness of asymptotics, see Lemma 1.7.

Fix $P \in \mathcal{P}^{\min}_\sigma$ with $P \subset Q$. Select a complete set $\mathcal{W}_{Q,v} \subset N_{K_Q}(\mathfrak{a}_\mathrm{q})$ of representatives for $W_Q / W_Q \cap W_{K \cap vHv^{-1}}$.

The set $\mathcal{W}_{Q,v}v$ maps injectively into the coset space $W/W_{K\cap H}$. Hence it may be extended to a complete set $\mathcal{W}$ of representatives in $N_K(\mathfrak{a}_\mathrm{q})$ for $W/W_{K\cap H}$. In view of Lemma 2.2 we may therefore decompose f, if necessary, so that we arrive in the situation that there exists a $s \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ such that $\mathrm{Exp}(P, uv \,|\, f) \subset s - \mathbb{N}\Delta(P)$, for all $u \in \mathcal{W}_{Q,v}$. We put $s_Q = s|\mathfrak{a}_{Q\mathrm{q}}$.

Let $u \in \mathcal{W}_{Q,v}$. Then the function $f_{uv}: a \mapsto f(auv)$ has a (unique) $\Delta(P)$-exponential polynomial expansion on $A_{\mathrm{q}}^+(P)$ of the following type:

$$f_{uv}(a) = f(auv) = \sum_{\xi \in s - \mathbb{N}\Delta(P)} q_{u,\xi}(\log a) a^\xi. \tag{3.10}$$

Here $q_{u,\xi}(\,\cdot\,) = q_\xi(P, uv \,|\, f, \,\cdot\,, e)$ belongs to $P_d(\mathfrak{a}_{\mathrm{q}}) \otimes V_\tau^{K_{\mathrm{M}} \cap uvHv^{-1}u^{-1}}$.

Let $\partial \in S(\mathfrak{a}_{\mathrm{q}})$. Then according to Lemma 1.9, the function ∂f_{uv} is given on $A_{\mathrm{q}}^+(P)$ by a neatly convergent $\Delta(P)$-exponential polynomial series that is obtained from (3.10) by term by term application of ∂. That is,

$$\partial f_{uv}(a) = \sum_{\xi \in s - \mathbb{N}\Delta(P)} q_{\partial,u,\xi}(\log a) a^\xi, \tag{3.11}$$

where $q_{\partial,u,\xi}$ is the $V_\tau^{K_{\mathrm{M}} \cap uvHv^{-1}u^{-1}}$-valued polynomial function on $\mathfrak{a}_{\mathrm{q}}$ of degree at most d given by

$$q_{\partial,u,\xi}(X) = e^{-\xi(X)} \partial[e^{\xi(\,\cdot\,)} q_{u,\xi}](X) \qquad (X \in \mathfrak{a}_{\mathrm{q}}).$$

Let now $R > 1$ and let $\mathcal{K}$ and $\mathcal{K}'$ be compact subsets of ${}^*A_{Q\mathrm{q}}^+({}^*P)_{[R]}$ and $A_{Q\mathrm{q}}^+(R^{-1})$, respectively. Then $\mathcal{K}'\mathcal{K}$ is a compact subset of $A_{\mathrm{q}}^+(P)$, by Lemma 3.3 (a). Thus, if $a \in \mathcal{K}'$ and $b \in \mathcal{K}$, then the series in (3.11) with ba in place of a converges absolutely, and may be rearranged as follows:

$$\partial f_{uv}(ab) = \sum_{\eta \in s_Q - \mathbb{N}\Delta_r(Q)} a^\eta \sum_{\substack{\xi \in s - \mathbb{N}\Delta(P) \\ \xi|\mathfrak{a}_{Q\mathrm{q}} = \eta}} b^\xi \, q_{\partial,u,\xi}(\log b + \log a). \tag{3.12}$$

In view of Lemma 1.5, the convergence is absolutely uniformly for $(a, b) \in \mathcal{K}' \times \mathcal{K}$.

By a similar reasoning it follows from the neat convergence of the series (3.11) that, for any continuous seminorm σ_0 on $P_d(\mathfrak{a}_{\mathrm{q}}) \otimes V_\tau$, the series

$$\sum_{\eta \in s_Q - \mathbb{N}\Delta_r(Q)} a^{\mathrm{Re}\,\eta} \sum_{\substack{\xi \in s - \mathbb{N}\Delta(P) \\ \xi|\mathfrak{a}_{Q\mathrm{q}} = \eta}} b^{\mathrm{Re}\,\xi} \, \sigma_0(q_{\partial,u,\xi}) \tag{3.13}$$

converges uniformly for $a \in \mathcal{K}'$ and $b \in \mathcal{K}$.

Let now $\eta \in s_Q - \mathbb{N}\Delta_r(Q)$ and let $b \in {}^*A_{Q\mathrm{q}}^+({}^*P)$ and $a \in A_{Q\mathrm{q}}^+(R_{Q,v}(b)^{-1})$. Then there exists a $R > 1$ such that $b \in {}^*A_{Q\mathrm{q}}^+({}^*P)_{[R]}$ and $a \in A_{Q\mathrm{q}}^+(R^{-1})$. Hence the series (3.13) converges, and by positivity of all of its terms we infer that the series

$$\sum_{\substack{\xi \in s - \mathbb{N}\Delta(P) \\ \xi|\mathfrak{a}_{Q\mathrm{q}} = \eta}} b^{\mathrm{Re}\,\xi} \, \sigma_0(q_{\partial,u,\xi}) \tag{3.14}$$

converges for every continuous seminorm σ_0 on $P_d(\mathfrak{a}_{\mathrm{q}}) \otimes V_\tau$, for every $b \in {}^*A_{Q\mathrm{q}}^+({}^*P)$.

We now specialize to $\partial = 1$ and note that $q_{1,u,\xi} = q_{u,\xi}$. Let $X \in \mathfrak{a}_{Q\mathrm{q}}$. We define the linear endomorphism T_X of $P_d(\mathfrak{a}_\mathrm{q}) \otimes V_\tau$ by $T_X p(H) = p(X + H)$. This endomorphism is continuous linear by finite dimensionality. Combining this with the convergence of (3.14) we infer, for every $X \in \mathfrak{a}_{Q\mathrm{q}}$, that

$$q_{Q,u,\eta}(X, b) := \sum_{\substack{\xi \in s - \mathbb{N}\Delta(P) \\ \xi\,|\,\mathfrak{a}_{Q\mathrm{q}} = \eta}} b^\xi\, T_X(q_{u,\xi})(\log b) \tag{3.15}$$

is a function of b defined by a neatly convergent $\Delta_Q(P)$-exponential polynomial series on ${}^*A^+_{Q\mathrm{q}}({}^*P)$. It is polynomial in X of degree at most d, and real analytic in $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$. Moreover, its values are in the space $V_\tau^{K_\mathrm{M} \cap uvHv^{-1}u^{-1}}$. Thus $q_{Q,u,\eta} \in P_d(\mathfrak{a}_{Q\mathrm{q}}) \otimes C^{\mathrm{ep}}({}^*A^+_{Q\mathrm{q}}({}^*P), V_\tau^{K_\mathrm{M} \cap uvHv^{-1}u^{-1}})$. In view of the isomorphism (2.9) for $\mathrm{X}_{Q,v,+}$, τ_Q, $\mathcal{W}_{Q,v}$ in place of X_+, τ, $\mathcal{W}$, see also the decomposition (3.5), there exists a unique polynomial function $q_\eta = q_\eta(Q, v\,|\,f)$ on $\mathfrak{a}_{Q\mathrm{q}}$ with values in $C^{\mathrm{ep}}(\mathrm{X}_{Q,v,+} : \tau_Q)$ such that

$$q_\eta(X, bu) = q_{Q,u,\eta}(X, b), \qquad (X \in \mathfrak{a}_{Q\mathrm{q}}, u \in \mathcal{W}_{Q,v}, b \in {}^*A^+_{Q\mathrm{q}}({}^*P)). \tag{3.16}$$

The degree of q_η as a polynomial function on $\mathfrak{a}_{Q\mathrm{q}}$ is at most d. Combining this with (3.15) and (3.12) and using that $R_{Q,v}(bu) = R_{Q,v}(b)$, we arrive at the expansion (3.9) for $m = bu$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$. Using the left K_Q-invariance of $R_{Q,v}$ and the sphericality of f and the functions $m \mapsto q_\eta(\log a, m)$, we now obtain (3.9) with absolute convergence; the first two assertions of (b) follow as well. The assertion of neat convergence in (a) is a consequence of the final assertion in (b), which we will now proceed to establish.

Let $u \in \mathcal{W}_{Q,v}$ and $R > 1$ be fixed. Then in view of the union (3.8) it suffices to prove the neat convergence of the series (3.9) as a $\Delta_r(Q)$-exponential polynomial series with coefficients in $C^\infty(K_Q{}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]}\, u(M_Q \cap vHv^{-1}) : \tau_Q)$. The map $(k, a) \mapsto kau(M_Q \cap vHv^{-1})$ induces a diffeomorphism from $K_Q/(K_Q \cap vHv^{-1}) \times {}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]}$ onto the open subset $K_Q{}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]}\, u(M_Q \cap vHv^{-1})$ of $\mathrm{X}_{Q,v,+}$. By sphericality of the coefficients of the series (3.9) we see that it suffices to prove that

$$\sum_{\eta \in s_Q - \mathbb{N}\Delta_r(Q)} a^\eta \sigma_1(q_{Q,u,\eta})$$

converges absolutely, for $a \in A^+_{Q\mathrm{q}}(R^{-1})$ and for σ_1 any continuous seminorm on $P_d(\mathfrak{a}_{Q\mathrm{q}}) \otimes C^\infty({}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]}, V_\tau^{K_\mathrm{M} \cap uvHv^{-1}u^{-1}})$.

Fix $X \in \mathfrak{a}_{Q\mathrm{q}}$, $\partial \in S({}^*\mathfrak{a}_{Q\mathrm{q}})$, $a \in A^+_{Q\mathrm{q}}(R^{-1})$ and $\mathcal{K} \subset {}^*A^+_{Q\mathrm{q}}({}^*P)_{[R]}$ a compact subset. Then it suffices to prove that

$$\sum_{\eta \in s_Q - \mathbb{N}\Delta_r(Q)} a^\eta \sup_{\mathcal{K}} \|\partial(q_{Q,u,\eta}(X, \cdot\,))\| \tag{3.17}$$

converges absolutely.

From the neat convergence of the series (3.15), for $b \in {}^*A_{Q\mathrm{q}}({}^*P)$, it follows that term by term differentiation is allowed. Since $\partial \in S({}^*\mathfrak{a}_{Q\mathrm{q}})$, whereas $X \in$

$\mathfrak{a}_{Q\mathrm{q}}$, we have

$$b^{-\xi}\partial(b^{\xi}T_X(q_{u,\xi})(\log b)) = q_{\partial,u,\xi}(X+\log b).$$

Hence, for every $\eta \in s_Q - \mathbb{N}\Delta_r(Q)$,

$$\partial(q_{Q,u,\eta}(X,\,\cdot\,))(b) = \sum_{\substack{\xi\in s-\mathbb{N}\Delta(P)\\ \xi|\mathfrak{a}_{Q\mathrm{q}}=\eta}} b^{\xi}q_{\partial,u,\xi}(X+\log b). \tag{3.18}$$

There exists a continuous seminorm σ_2 on $P_d(\mathfrak{a}_\mathrm{q})\otimes V_\tau$, such that, for every $b\in\mathcal{K}$ and all $q\in P_d(\mathfrak{a}_\mathrm{q})\otimes V_\tau$,

$$\|q(X+\log b)\| \le \sigma_2(q).$$

In particular, this implies that

$$\|q_{\partial,u,\xi}(X+\log b)\| \le \sigma_2(q_{\partial,u,\xi}), \tag{3.19}$$

for every $b\in\mathcal{K}$.

Combining (3.18) with (3.19) we now obtain

$$|a^{\eta}|\sup_{\mathcal{K}}\|\partial(q_{Q,u,\eta})(X,\,\cdot\,)\| \le \sum_{\substack{\xi\in s-\mathbb{N}\Delta(P)\\ \xi|\mathfrak{a}_{Q\mathrm{q}}=\eta}} a^{\mathrm{Re}\,\eta}b^{\mathrm{Re}\,\xi}\,\sigma_2(q_{\partial,u,\xi}).$$

Thus, the absolute convergence of (3.17) follows from the uniform convergence of (3.13), $b\in\mathcal{K}$. □

Let $f\in C^{\mathrm{ep}}(\mathrm{X}_+:\tau)$ and let $Q\in\mathcal{P}_\sigma$ and $v\in N_K(\mathfrak{a}_\mathrm{q})$. Moreover, let the set $Y\subset\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and the polynomials $q_\eta = q_\eta(Q,v\,|\,f)$, for $\eta\in Y-\mathbb{N}\Delta_r(Q)$ be as in Theorem 3.4. As in that theorem, we define

$$\mathrm{Exp}(Q,v\,|\,f) = \{\eta\in Y-\mathbb{N}\Delta_r(Q)\mid q_\eta\neq 0\}$$

and call the elements of this set the exponents of f along (Q,v). If $\eta\in\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ does not belong to $\mathrm{Exp}(Q,v\,|\,f)$, we agree to write $q_\eta(Q,v\,|\,f)=0$.

Let now $P\in\mathcal{P}_\sigma^{\min}$ be contained in Q and put ${}^*P{:}=P\cap M_Q$. Then, for $u\in N_{K_Q}(\mathfrak{a}_\mathrm{q})$, we define

$$\mathrm{Exp}(Q,v\,|\,f)_{P,u} = \{\eta\in\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}\mid q_\eta\neq 0\quad\text{on}\quad \mathfrak{a}_{Q\mathrm{q}}\times K_Q\,{}^*A^+_{Q\mathrm{q}}({}^*P)u(M_Q\cap vHv^{-1})\}.$$

The elements of this set are called the (Q,v)-exponents of f on ${}^*A^+_{Q\mathrm{q}}({}^*P)u$. Let $\mathcal{W}_{Q,v}\subset N_{K_Q}(\mathfrak{a}_\mathrm{q})$ be a complete set of representatives of $W_Q/W_Q\cap W_{K\cap vHv^{-1}}$. Then it follows from (3.5) that

$$\mathrm{Exp}(Q,v\,|\,f) = \bigcup_{u\in\mathcal{W}_{Q,v}}\mathrm{Exp}(Q,v\,|\,f)_{P,u}. \tag{3.20}$$

We now have the following result.

Theorem 3.5 (Transitivity of asymptotics) *Let $f \in C^{\rm ep}(\mathrm{X}_+ : \tau)$. Let $P, Q \in \mathcal{P}_\sigma$, assume that P is minimal and $P \subset Q$ and put ${}^*P = P \cap M_Q$. Then for all $v \in N_K(\mathfrak{a}_{\rm q})$ and $u \in N_{K_Q}(\mathfrak{a}_{\rm q})$ we have:*

$$\mathrm{Exp}(Q, v \,|\, f)_{P,u} = \mathrm{Exp}(P, uv \,|\, f)\,|_{\mathfrak{a}_{Q\rm q}}. \tag{3.21}$$

Moreover, if $\eta \in \mathrm{Exp}(P, uv \,|\, f)|_{\mathfrak{a}_{Q\rm q}}$, then for every $b \in {}^*A^+_{Q\rm q}({}^*P)$, $X \in \mathfrak{a}_{Q\rm q}$, and $m \in M$,

$$q_\eta(Q, v \,|\, f, X, mbu) = \sum_{\substack{\xi \in \mathrm{Exp}(P, uv \,|\, f) \\ \xi|_{\mathfrak{a}_{Q\rm q}} = \eta}} b^\xi q_\xi(P, uv \,|\, f, X + \log b, m), \tag{3.22}$$

where the $\Delta_Q(P)$-exponential polynomial series (in the variable b) on the right is neatly convergent on ${}^*A^+_{Q\rm q}({}^*P)$. Furthermore, the series

$$\sum_{\substack{\xi \in \mathrm{Exp}(P, uv \,|\, f) \\ \xi|_{\mathfrak{a}_{Q\rm q}} = \eta}} b^\xi q_\xi(P, uv \,|\, f, X + \log b) \tag{3.23}$$

converges neatly as a $\Delta_Q(P)$-exponential polynomial series in the variable $b \in {}^*A^+_{Q\rm q}({}^*P)$ with coefficients in $C^\infty(\mathrm{X}_{0,uv} : \tau_{\rm M})$.

Proof: Let $v \in N_K(\mathfrak{a}_{\rm q})$ and $u \in N_{K_Q}(\mathfrak{a}_{\rm q})$ be fixed. Fix a set $\mathcal{W}_{Q,v}$ such as in the beginning of the proof of Theorem 3.4, and such that it contains u. Moreover, we select a set $\mathcal{W}$ of representatives for $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_{\rm q})$ containing $\mathcal{W}_{Q,v}v$. As in the proof of the mentioned theorem we may restrict ourselves to the situation that $\mathrm{Exp}(P, u'v \,|\, f) \subset s - \mathbb{N}\Delta(P)$, for some $s \in \mathfrak{a}^*_{\rm q\mathbb{C}}$ and all $u' \in \mathcal{W}_{Q,v}$. In the following we may now use the notation and results of the proof of Theorem 3.4.

Let $\eta \in s_Q - \mathbb{N}\Delta_r(Q)$. Then from (3.16) and (3.15) we infer that, for every $X \in \mathfrak{a}_{Q\rm q}$,

$$q_\eta(Q, v \,|\, f, X, bu) = \sum_{\substack{\xi \in s - \mathbb{N}\Delta(P) \\ \xi|_{\mathfrak{a}_{Q\rm q}} = \eta}} b^\xi q_\xi(P, uv \,|\, f, X + \log b, e), \qquad (b \in {}^*A^+_{Q\rm q}({}^*P));$$

the series on the left-hand side converges neatly as a $\Delta_Q(P)$-exponential polynomial series in the variable $b \in {}^*A^+_{Q\rm q}({}^*P)$. The function $m \mapsto q_\eta(Q, v \,|\, f, X, mbu)$ belongs to $C^\infty(\mathrm{X}_{0,uv} : \tau_{\rm M})$, and so does the function $m \mapsto q_\xi(P, uv \,|\, f, X + \log b, m)$, for every $\xi \in s - \mathbb{N}\Delta(P)$. Evaluation at e induces a topological linear isomorphism $C^\infty(X_{0,w} : \tau_{\rm M}) \simeq V_\tau^{M \cap wHw^{-1}}$, for every $w \in N_K(\mathfrak{a}_{\rm q})$, hence in particular for $w = uv$. Thus, it follows from the above that (3.22) holds, with the asserted convergence. In addition, it follows that the series (3.23) converges as asserted.

In the proof of Theorem 3.4 we saw that $\mathrm{Exp}(Q, v \,|\, f) \subset s_Q - \mathbb{N}\Delta_r(Q)$. It follows from the derived expansion (3.22) that (3.21) holds with the inclusion '$\subset$' in place of the equality sign. For the converse inclusion, let $\xi_0 \in \mathrm{Exp}(P, uv \,|\, f)$ and put $\eta = \xi_0|_{\mathfrak{a}_{Q\rm q}}$. We select $X \in \mathfrak{a}_{Q\rm q}$ such that the function $b \mapsto q_{\xi_0}(P, uv \,|\, f, X + \log b, e)$ does not vanish identically on ${}^*A_{Q\rm q}$. The equality

(3.22) holds for all $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$ with a $\Delta_Q(P)$-exponential polynomial series that converges neatly on ${}^*A^+_{Q\mathrm{q}}({}^*P)$. Any exponent ξ of this series coincides with $\eta = \xi_0|_{\mathfrak{a}_{Q\mathrm{q}}}$ on $\mathfrak{a}_{Q\mathrm{q}}$; if it also coincides with ξ_0 on ${}^*\mathfrak{a}_{Q\mathrm{q}}$, then $\xi = \xi_0$. Therefore, the function of b defined by the series on the right-hand side of (3.22) is non-zero. Hence $q_\eta(Q, v \,|\, f)$ does not vanish identically on $\mathfrak{a}_{Q\mathrm{q}} \times {}^*A^+_{Q\mathrm{q}}({}^*P)u$ and we conclude that $\eta \in \mathrm{Exp}(Q, v \,|\, f)_{P,u}$. $\square$

We proceed by discussing some useful transformation properties for the coefficients in the expansion (3.9).

If $u \in N_K(\mathfrak{a}_\mathrm{q})$ it will sometimes be convenient to write $uX := \mathrm{Ad}(u)X$ for $X \in \mathfrak{a}_\mathrm{q}$. Similarly, we will write $u\xi := \xi \circ \mathrm{Ad}(u)^{-1}$, for $\xi \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

If $u, v \in N_K(\mathfrak{a}_\mathrm{q})$ and $Q \in \mathcal{P}_\sigma$, then conjugation by u induces a diffeomorphism γ_u from the space $\mathrm{X}_{Q,v}$ onto $\mathrm{X}_{uQu^{-1},uv}$; we note that γ_u maps $\mathrm{X}_{Q,v,+}$ onto $\mathrm{X}_{uQu^{-1},uv,+}$. It is easily seen that $R_{uQu^{-1},uv}(\gamma_u(m)) = R_{Q,v}(m)$, for $m \in \mathrm{X}_{Q,v}$.

For $\varphi \in C^\infty(\mathrm{X}_{Q,v,+} : \tau_Q)$, we define the function $\rho_{\tau,u}\varphi \colon \mathrm{X}_{uQu^{-1},uv,+} \to V_\tau$ by

$$\rho_{\tau,u}\varphi(x) = \tau(u)\varphi(\gamma_u^{-1}(x)). \tag{3.24}$$

Then $\rho_{\tau,u}$ is a topological linear isomorphism from the space $C^\infty(\mathrm{X}_{Q,v,+} : \tau_Q)$ onto the space $C^\infty(\mathrm{X}_{uQu^{-1},uv,+} : \tau_{uQu^{-1}})$. Likewise, by similar definitions we obtain a topological linear isomorphism from $C^\infty(\mathrm{X}_{1Q,v,+} : \tau_Q)$ onto $C^\infty(\mathrm{X}_{1\,uQu^{-1},uv,+} : \tau_{uQu^{-1}})$, also denoted by $\rho_{\tau,u}$.

Lemma 3.6 *Let $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, let $Q \in \mathcal{P}_\sigma$ and $u, v \in N_K(\mathfrak{a}_\mathrm{q})$. Then*

$$\mathrm{Exp}(uQu^{-1}, uv \,|\, f) = u\,\mathrm{Exp}(Q, v \,|\, f).$$

Moreover, for every $\eta \in \mathrm{Exp}(Q, v \,|\, f)$,

$$q_{u\eta}(uQu^{-1}, uv \,|\, f) = [\mathrm{Ad}(u^{-1})^* \otimes \rho_{\tau,u}]\, q_\eta(Q, v \,|\, f).$$

Proof: Put $Q' = uQu^{-1}$. Let $m \in \mathrm{X}_{Q',uv,+}$. Then, by Theorem 3.4,

$$f(mauv) = \sum_{\eta \in \mathrm{Exp}(Q', uv \,|\, f)} a^\eta q_\eta(Q', uv \,|\, f)(\log a, m), \tag{3.25}$$

for $a \in A^+_{Q'\mathrm{q}}(R_{Q',uv}(m)^{-1})$, where the series on the right-hand side is neatly convergent. On the other hand, from $f(mauv) = \tau(u)f(\gamma_u^{-1}(m)u^{-1}au\,v)$ we see, using Theorem 3.4 again, that

$$f(mauv) = \tau(u) \sum_{\zeta \in \mathrm{Exp}(Q, v \,|\, f)} a^{u\zeta} q_\zeta(Q, v \,|\, f)(\mathrm{Ad}(u)^{-1}\log a, \gamma_u^{-1}(m)), \tag{3.26}$$

for $u^{-1}au \in A^+_{Q\mathrm{q}}(R_{Q,v}(\gamma_u^{-1}(m))^{-1})$. We now note that the latter condition is equivalent to

$$a \in A^+_{Q'\mathrm{q}}(R_{Q,v}(\gamma_u^{-1}(m))^{-1}) = A^+_{Q'\mathrm{q}}(R_{Q',uv}(m)^{-1}).$$

Hence the series (3.25) and (3.26) both converge neatly for $a \in A_{Q'\mathrm{q}}(R_{Q',uv}(m)^{-1})$. All assertions now follow by uniqueness of asymptotics. $\square$

For later purposes, we also need another type of transformation property. Recall from Remark 3.1 that for $u \in N_K(\mathfrak{a}_q)$ we write $X_u = X_{1G,u} = G/uHu^{-1}$; let $X_{u,+}$ denote the analogue of X_+ for this symmetric space. We note that right multiplication by u induces a diffeomorphism r_u from X_u onto X, mapping $X_{u,+}$ onto X_+. Hence pull-back by r_u the topological linear isomorphism $R_u := r_u^*$ from $C^\infty(X_+ : \tau)$ onto $C^\infty(X_{u,+} : \tau)$; it is given by $R_u f(x) = f(xu)$. We note that the map R_u coincides with the map $\rho_{\tau,u}$, introduced in the text above Lemma 3.6, by sphericality of the functions involved.

The following result is now an immediate consequence of the definitions.

Lemma 3.7 *Let $f \in C^{\rm ep}(X_+ : \tau)$ and $u \in N_K(\mathfrak{a}_q)$. Then $R_u f \in C^{\rm ep}(X_{u,+} : \tau)$. Moreover, for each $Q \in \mathcal{P}_\sigma$ and every $v \in N_K(\mathfrak{a}_q)$, the set $\mathrm{Exp}(Q, vu \,|\, f)$ equals $\mathrm{Exp}(Q, v \,|\, R_u f)$. Finally, if $\xi \in \mathrm{Exp}(Q, vu \,|\, f)$, then*

$$q_\xi(Q, vu \,|\, f) = q_\xi(Q, v \,|\, R_u f).$$

4 Behavior of differential operators along walls

We assume that $Q \in \mathcal{P}_\sigma$ is fixed. The purpose of this section is to study a Q-radial decomposition of invariant differential operators on X. This leads to a series expansion of such operators along (Q, e), with coefficients that turn out to be globally defined on the group $M_{Q\sigma}$. This will be of crucial importance for the applications later on (see Proposition 8.3).

The involution $\theta\sigma$ fixes $\mathfrak{a}_q$ pointwise, hence leaves every root space $\mathfrak{g}_\alpha$, for $\alpha \in \Sigma$, invariant. We denote the associated eigenspaces of $\theta\sigma|_{\mathfrak{g}_\alpha}$ for the eigenvalues $+1$ and -1 by $\mathfrak{g}_\alpha^+$ and $\mathfrak{g}_\alpha^-$, respectively. Moreover, we put $m_\alpha^\pm := \dim \mathfrak{g}_\alpha^\pm$.

We recall that $K_Q = K \cap M_Q$ and $H_Q = H \cap M_Q$. Define $H_{1Q} := H \cap M_{1Q}$; then $H_{1Q} = H_Q(A_Q \cap H)$. Note that $K_Q = K \cap M_{1Q}$. The group M_{1Q} admits the Cartan decomposition $M_{1Q} = K_Q A_q H_{1Q}$ and normalizes the subalgebra $\bar{\mathfrak{n}}_Q$.

For $m \in M_{1Q}$ we define the endomorphism $A(m) = A_Q(m) \in \mathrm{End}(\bar{\mathfrak{n}}_Q)$ by

$$A(m) := \sigma \circ \mathrm{Ad}(m^{-1}) \circ \theta \circ \mathrm{Ad}(m). \tag{4.1}$$

Moreover, we define the real analytic function $\delta = \delta_Q : M_{1Q} \to \mathbb{R}$ by

$$\delta(m) = \det(I - A(m)). \tag{4.2}$$

Finally, we define the following subset of M_{1Q}

$$M'_{1Q} := M_{1Q} \setminus \delta^{-1}(0). \tag{4.3}$$

Lemma 4.1

(a) *Let $m \in M_{1Q}$, $k \in K_Q$ and $h \in H_{1Q}$. Then $A(kmh) = \mathrm{Ad}(h^{-1}) \circ A(m) \circ \mathrm{Ad}(h)$.*

(b) *The endomorphism* $A(m) \in \mathrm{End}(\bar{\mathfrak{n}}_Q)$ *is diagonalizable, for every* $m \in M_{1Q}$*. The eigenvalues are given as follows. Let* $m = kah$*, with* $k \in K_Q$*,* $a \in A_{\mathrm{q}}$ *and* $h \in H_{1Q}$*. Then the eigenvalues of* $A(m)$ *are* $\pm a^{-2\alpha}$*,* $\alpha \in \Sigma(Q)$*, with multiplicities* $m_\alpha^\pm$*.*

(c) *The operator norm of* $A(m)$ *is given by* $\|A(m)\|_{\mathrm{op}} = R_{Q,1}(m)^2$*.*

Proof: (a) is an immediate consequence of (4.1). Hence, for (b) we may assume that $m = a \in A_{\mathrm{q}}$. It is easily seen that $A(a)|_{\mathfrak{g}^\pm_{-\alpha}} = \pm a^{-2\alpha} I$ for $\alpha \in \Sigma(Q)$.

Finally, (c) is an immediate consequence of (b) and (3.6) with $v = 1$. □

Corollary 4.2 *If* $k \in K_Q$*,* $a \in A_{\mathrm{q}}$*,* $h \in H_{1Q}$ *then*

$$\delta(kah) = \prod_{\alpha \in \Sigma(Q)} (1 - a^{-2\alpha})^{m_\alpha^+} (1 + a^{-2\alpha})^{m_\alpha^-}.$$

The set M'_{1Q} *is left* K_Q*- and right* H_{1Q}*-invariant, and open dense in* M_{1Q}*.*

Proof: This follows immediately from Lemma 4.1 combined with (4.2) and (4.3). □

We define the linear subspace $\mathfrak{k}(Q)$ of $\mathfrak{k}$ by $\mathfrak{k}(Q) := \mathfrak{k} \cap (\mathfrak{n}_Q + \bar{\mathfrak{n}}_Q)$. Then the map $(I + \theta): X \mapsto X + \theta X$ is a linear isomorphism from $\bar{\mathfrak{n}}_Q$ onto $\mathfrak{k}(Q)$.

Lemma 4.3

(a) *If* $m \in M_{1Q}$*, then* $\mathrm{Ad}(m^{-1})\mathfrak{k}(Q) + \mathfrak{h} \subset \bar{\mathfrak{n}}_Q + \mathfrak{h}$*.*

(b) *If* $m \in M'_{1Q}$*, then* $\mathrm{Ad}(m^{-1})\mathfrak{k}(Q) \oplus \mathfrak{h} = \bar{\mathfrak{n}}_Q + \mathfrak{h}$*.*

Proof: (a) Since $\mathfrak{k}(Q) \subset \bar{\mathfrak{n}}_Q + \mathfrak{n}_Q \subset \bar{\mathfrak{n}}_Q + \mathfrak{h}$, we have, for all $m \in M_{1Q}$,

$$\mathrm{Ad}(m^{-1})\mathfrak{k}(Q) \subset \mathrm{Ad}(m^{-1})(\bar{\mathfrak{n}}_Q + \mathfrak{n}_Q) = \bar{\mathfrak{n}}_Q + \mathfrak{n}_Q \subset \bar{\mathfrak{n}}_Q + \mathfrak{h}.$$

(b) The dimension of $\mathrm{Ad}(m^{-1})\mathfrak{k}(Q)$ equals that of $\mathfrak{k}(Q)$, which in turn equals that of $\bar{\mathfrak{n}}_Q$. Hence it suffices to prove, for $m \in M'_{1Q}$, that $\mathrm{Ad}(m^{-1})\mathfrak{k}(Q) \cap \mathfrak{h} = 0$.

Let $X \in \mathrm{Ad}(m^{-1})\mathfrak{k}(Q) \cap \mathfrak{h}$. Then $\theta \mathrm{Ad}(m) X = \mathrm{Ad}(m) X$ and $\sigma X = X$, and we see that $(I - A(m))X = 0$. If $m \in M'_{1Q}$ then $\det(I - \mathrm{Ad}(m)) = \delta(m) \neq 0$ and it follows that $X = 0$. □

From Lemma 4.3(b) we see that for $m \in M'_{1Q}$ we may define linear maps $\Psi(m) = \Psi_Q(m) \in \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{k}(Q))$ and $R(m) = R_Q(m) \in \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{h})$ by

$$X = \mathrm{Ad}(m^{-1})\Psi(m)X + R(m)X. \tag{4.4}$$

Lemma 4.4 *Let $m \in M'_{1Q}$, $k \in K_Q$ and $h \in H_{1Q}$. Then*

$$\begin{aligned}\Psi(kmh) &= \mathrm{Ad}(k) \circ \Psi(m) \circ \mathrm{Ad}(h), \\ R(kmh) &= \mathrm{Ad}(h^{-1}) \circ R(m) \circ \mathrm{Ad}(h).\end{aligned}$$

Proof: This is an immediate consequence of (4.4) combined with Lemma 4.3(b). □

Lemma 4.5 *Let $m \in M'_{1Q}$. Then*

$$\begin{aligned}\Psi(m) \circ (I - A(m)) &= (I + \theta) \circ \mathrm{Ad}(m), \\ R(m) \circ (I - A(m)) &= -(I + \sigma) \circ A(m).\end{aligned}$$

Proof: From (4.1) it follows that

$$I + \sigma \circ A(m) = \mathrm{Ad}(m^{-1}) \circ (I + \theta) \circ \mathrm{Ad}(m).$$

This implies in turn that

$$I - A(m) = \mathrm{Ad}(m^{-1}) \circ (I + \theta) \circ \mathrm{Ad}(m) - (I + \sigma) \circ A(m). \tag{4.5}$$

Since $I + \theta$ and $I + \sigma$ map $\bar{\mathfrak{n}}_Q$ into $\mathfrak{k}(Q)$ and $\mathfrak{h}$, respectively, the lemma follows from combining (4.5) with (4.4). □

Corollary 4.6 *The functions $\Psi \colon M'_{1Q} \to \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{k}(Q))$ and $R \colon M'_{1Q} \to \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{h})$ are real analytic. Moreover, the functions $\delta\, \Psi$ and $\delta\, R$ extend to real analytic functions on M_{1Q}.*

Proof: From (4.2) and (4.3) we see that $I - A(m)$ is an invertible endomorphism of $\bar{\mathfrak{n}}_Q$, for $m \in M'_{1Q}$. Since $\mathrm{Ad}(m)$ and $A(m)$ depend real analytically on $m \in M_{1Q}$, all statements now follow from Lemma 4.5. □

If $R > 0$, then in accordance with (3.7) we define

$$M_{1Q}[R] \colon = \{m \in M_{1Q} \mid R_{Q,1}(m) < R\}.$$

Moreover, we set $M_{Q\sigma}[R] \colon = M_{Q\sigma} \cap M_{1Q}[R]$.

Lemma 4.7

(a) $M_{1Q}[1] \subset M'_{1Q}$.

(b) *Let $R_1, R_2 > 0$. Then $M_{Q\sigma}[R_1]\, A^+_{Q\mathrm{q}}(R_2) \subset M_{1Q}[R_1 R_2]$.*

Proof: Let $m \in M_{1Q}[1]$. Then $\|A(m)\|_{\mathrm{op}} < 1$ by Lemma 4.1(c), and hence $\delta(m) \neq 0$. This establishes (a).

Assume that $m \in M_{Q\sigma}[R_1]$ and $a \in A^+_{Q\mathrm{q}}(R_2)$. Write $m = kbh$ with $k \in K_Q$, $b \in {}^*A_{Q\mathrm{q}}$ and $h \in H_{1Q}$. Then $ma = k(ab)h$, hence $R_{Q,1}(ma) = \max_{\alpha \in \Sigma(Q)} a^{-\alpha} b^{-\alpha} < R_2 R_{Q,1}(m) < R_1 R_2$. It follows that $ma \in M_{1Q}[R_1 R_2]$. □

Proposition 4.8 *There exist unique real analytic functions $\Psi_\mu, R_\mu\colon M_{Q\sigma} \to \mathrm{End}(\bar{\mathfrak{n}}_Q)$, for $\mu \in \mathbb{N}\Delta_r(Q)$, such that for every $m \in M_{Q\sigma}$ and every $a \in A^+_{Q\mathrm{q}}(R_{Q,1}(m)^{-1})$,*

$$\begin{aligned}\Psi(ma) &= (1+\theta)\circ \sum_{\mu\in\mathbb{N}\Delta_r(Q)} a^{-\mu}\Psi_\mu(m),\\ R(ma) &= (1+\sigma)\circ \sum_{\mu\in\mathbb{N}\Delta_r(Q)} a^{-\mu}R_\mu(m),\end{aligned}$$

with absolutely convergent series. For every $R > 1$ the above series converge neatly on $A^+_{Q\mathrm{q}}(R^{-1})$ as $\Delta_r(Q)$-power series with coefficients in $C^\infty(M_{Q\sigma}[R], \mathrm{End}(\bar{\mathfrak{n}}_Q))$.

Proof: Let $m \in M_{Q\sigma}$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,1}(m)^{-1})$. It follows from Lemma 4.7 that $ma \in M_{1Q}[1] \subset M'_{1Q}$. Hence $\Psi(ma)$ and $R(ma)$ are defined.

It follows from Lemma 4.1 that $\|A(ma)\|_{\mathrm{op}} < 1$. Hence the series

$$(I - A(ma))^{-1} = \sum_{n=0}^{\infty} A(ma)^n$$

converges absolutely. Let $\alpha \in \Sigma_r(Q)$. Then $A(m)$ leaves the space $\mathfrak{g}_{-\alpha}$ invariant, and $A(ma)|_{\mathfrak{g}_{-\alpha}} = a^{-2\alpha}A(m)|_{\mathfrak{g}_{-\alpha}}$. Hence, in view of Lemma 4.5,

$$\Psi(ma)|_{\mathfrak{g}_{-\alpha}} = (I+\theta)\circ \mathrm{Ad}(m)\circ \sum_{n=0}^{\infty} a^{-(2n+1)\alpha}A(m)^n|_{\mathfrak{g}_{-\alpha}}$$

and

$$R(ma)|_{\mathfrak{g}_{-\alpha}} = -(I+\sigma)\circ \sum_{n=1}^{\infty} a^{-2n\alpha}A(m)^n|_{\mathfrak{g}_{-\alpha}}.$$

It is now easy to complete the proof. □

We denote by $\mathcal{R}^+_Q$ the algebra of functions on M'_{1Q} generated by the functions $\xi\circ\Psi_Q$, where $\xi \in \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{k}_{(Q)})^*$, and by the functions $\eta\circ \mathrm{R}_Q$, where $\eta \in \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{h})^*$. By $\mathcal{R}_Q$ we denote the algebra of functions generated by 1 and $\mathcal{R}^+_Q$. Note that $\mathcal{R}^+_Q$ is an ideal in $\mathcal{R}_Q$.

Corollary 4.9 *The elements of $\mathcal{R}_Q$ are left K_Q- and right H_{1Q}-finite functions on M'_{1Q}.*

Let $\varphi \in \mathcal{R}_Q$. There exists a $k \in \mathbb{N}$ such that $\delta_Q^k\varphi$ extends to a real analytic function on M_{1Q}. Moreover, there exist unique real analytic functions φ_ξ on $M_{Q\sigma}$, for $\xi \in \mathbb{N}\Delta_r(Q)$, such that for every $m \in M_{Q\sigma}$ and every $a \in A_{Q\mathrm{q}}(R_{Q,1}(m)^{-1})$,

$$\varphi(ma) = \sum_{\xi\in\mathbb{N}\Delta_r(Q)} a^{-\xi}\varphi_\xi(m). \tag{4.6}$$

Let $R \geq 1$. Then the series (4.6) converges neatly on $A_{Q\mathrm{q}}(R^{-1})$, as an exponential polynomial series with coefficients in $C^\infty(M_{Q\sigma}[R])$.

Finally, if $\varphi \in \mathcal{R}_Q^+$, then (4.6) holds with $\varphi_0 = 0$.

Proof: Uniqueness of the functions φ_ξ is obvious. Therefore it suffices to prove existence and the remaining assertions. One readily checks that it suffices to prove the assertions for a collection of generators of the algebra $\mathcal{R}_Q^+$. Such a collection of generators is formed by the functions of the form $\varphi = \xi \circ \Psi$, with $\xi \in \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{k}_{(Q)})^*$, and by the functions of the form $\varphi = \eta \circ \mathrm{R}_Q$, where $\eta \in \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{h})^*$. For both types of generators all assertions follow immediately from Proposition 4.8. □

As is the previous section we assume that τ is a smooth representation of K in a locally convex space V_τ. The space of continuous linear endomorphisms of V_τ is denoted by $\mathrm{End}(V_\tau)$.

If an element u of the space

$$\mathcal{D}_{1Q} := \mathcal{R}_Q \otimes \mathrm{End}(V_\tau) \otimes U(\mathfrak{m}_{1Q}) \tag{4.7}$$

is of the form $\varphi \otimes L \otimes v$, with $\varphi \in \mathcal{R}_Q$, $L \in \mathrm{End}(V_\tau)$, and $v \in U(\mathfrak{m}_{1Q})$, then we define the differential operator u_* on $C^\infty(M'_{1Q}, V_\tau)$ by $u_* f = \varphi L \circ [R_v f]$; here R denotes the right regular representation. The map $u \mapsto u_*$ extends to an injective linear map from $\mathcal{D}_{1Q}$ to the space of smooth $\mathrm{End}(V_\tau)$-valued differential operators of $C^\infty(M'_{1Q}, V_\tau)$. We also define the subspace

$$\mathcal{D}_{1Q}^+ := \mathcal{R}_Q^+ \otimes \mathrm{End}(V_\tau) \otimes U(\mathfrak{m}_{1Q}).$$

Via the map $u \mapsto 1 \otimes I \otimes u$ we identify $U(\mathfrak{m}_{1Q})$ with a subspace of $\mathcal{D}_{1Q}$. Then $u_* = R_u$ for $u \in U(\mathfrak{m}_{1Q})$.

Let $M_{1Q,+}$ be the preimage in M_{1Q} of the set $\mathrm{X}_{1Q,1,+}$ (see below (3.5)). The set

$$M'_{1Q,+} := M_{1Q,+} \cap M'_{1Q}$$

is an open dense subset of M_{1Q} that is left K_Q- and right H_{1Q}-invariant.

In view of the decomposition $\mathfrak{g} = \bar{\mathfrak{n}}_Q \oplus (\mathfrak{m}_{1Q} + \mathfrak{h})$, there exists, for every $D \in U(\mathfrak{g})$, an element $D_0 \in U(\mathfrak{m}_{1Q})$ with $\deg(D_0) \leq \deg(D)$, such that

$$D - D_0 \in \bar{\mathfrak{n}}_Q U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}. \tag{4.8}$$

The element D_0 is uniquely determined modulo $U(\mathfrak{m}_{1Q})\mathfrak{h}_{1Q}$. We recall from [5], Sect. 2, see also [7], p. 548-549, that the assignment $D \mapsto D_0$ induces an algebra homomorphism $\mu'_Q = {}^{\backprime}\mu_{\bar{Q}}: \mathbb{D}(\mathrm{X}) \to \mathbb{D}(M_{1Q}/H_{1Q})$, and that the homomorphism $\mu_Q: \mathbb{D}(\mathrm{X}) \to \mathbb{D}(M_{1Q}/H_{1Q})$, defined by $\mu_Q(D) = d_Q \circ \mu'_Q(D) \circ d_Q^{-1}$ with $d_Q(m) := |\det(\mathrm{Ad}(m)|_{\mathfrak{n}_Q})|^{1/2}$ for $m \in M_{1Q}$, only depends on Q through the Levi component M_{1Q}.

Proposition 4.10 *Let $D \in \mathbb{D}(\mathrm{X})$. There exists a $u_+ \in \mathcal{D}_{1Q}^+$ of degree $\deg(u_+) < \deg(D)$ such that, for every $f \in C^\infty(\mathrm{X}_+ : \tau)$,*

$$Df|_{M'_{1Q,+}} = [\mu'_Q(D) + u_{+*}](f|_{M'_{1Q,+}}).$$

Proof: By induction on the degree we will first establish the following assertion for an element D of $U(\mathfrak{g})$. Let $D_0 \in U(\mathfrak{m}_{1Q})$ satisfy (4.8). Then there exist finitely many $\varphi_i \in \mathcal{R}_Q^+$, $u_i \in U(\mathfrak{k})$, and $v_i \in U(\mathfrak{m}_{1Q})$, for $1 \le i \le n$, such that $\deg(u_i) + \deg(v_i) < \deg(D)$, and such that

$$D - D_0 \equiv \sum_{i=1}^{n} \varphi_i(m)\,[\mathrm{Ad}(m)^{-1}u_i]\,v_i \qquad \mathrm{mod} \quad U(\mathfrak{g})\mathfrak{h}, \tag{4.9}$$

for every $m \in M'_{1Q}$.

The assertion is trivially true for D constant. Thus, assume that D is not constant and that the assertion has been established for D of strictly smaller degree. Let $D_0 \in U(\mathfrak{m}_{1Q})$ be as above. Then, modulo $U(\mathfrak{g})\mathfrak{h}$, $D - D_0$ equals a finite sum of terms of the form XD_1, with $X \in \bar{\mathfrak{n}}_Q$ and $D_1 \in U(\bar{\mathfrak{n}}_Q \oplus \mathfrak{m}_{1Q})$ such that $\deg D_1 < \deg D$.

For $m \in M'_{1Q,+}$ we have $X = \mathrm{Ad}(m)^{-1}\Psi(m)X + \mathrm{R}_Q(m)X$; hence

$$XD_1 \equiv (\mathrm{Ad}(m)^{-1}\Psi(m)X)D_1 + [\mathrm{R}_Q(m)X, D_1] \qquad \mathrm{mod} \quad U(\mathfrak{g})\mathfrak{h}.$$

Now $\mathrm{Ad}(m)^{-1}\Psi(m)X$ is a finite sum of terms of the form $\varphi(m)[\mathrm{Ad}(m)^{-1}u]$ with $u \in \mathfrak{k}_{(Q)}$ and $\varphi \in \mathcal{R}_Q^+$. Applying the induction hypothesis to D_1 we see that $[\mathrm{Ad}(m)^{-1}\Psi(m)X]D_1$ may be expressed as a sum similar to the one on the right-hand side of (4.9).

On the other hand, $[\mathrm{R}_Q(m)X, D_1]$ is a finite sum of elements of the form $\psi(m)D_2$, with $\psi \in \mathcal{R}_Q^+$ and $D_2 \in U(\mathfrak{g})$, $\deg D_2 < \deg D$. Applying the induction hypothesis to D_2, we see that $[\mathrm{R}_Q(m)X, D_1]$ may also be expressed as a sum of the form (4.9). This establishes the assertion involving (4.9) of the beginning of the proof.

Let now $D \in \mathbb{D}(\mathrm{X})$. By abuse of notation we use the same symbol D for a representative of D in $U(\mathfrak{g})^H$, and let D_0 be as above. Then $\mu'_Q(D)$ equals the canonical image of D_0 in $U(\mathfrak{m}_{1Q})^{H_{1Q}}$. Let φ_i, u_i, v_i be as above and such that (4.9) holds. Then for every $f \in C^\infty(\mathrm{X}_+ : \tau)$ and all $m \in M'_{1Q,+}$ we have

$$\begin{aligned} Df(m) &= \mu'_Q(D)(f|_{M'_{1Q,+}})(m) + \sum_{i=1}^{n} \varphi_i(m) R_{\mathrm{Ad}(m)^{-1}u_i} R_{v_i} f(m) \\ &= \mu'_Q(D)(f|_{M'_{1Q,+}})(m) + \sum_{i=1}^{n} \varphi_i(m) \tau(\check{u}_i) R_{v_i} f(m) \end{aligned}$$

where we have used that $R_{\mathrm{Ad}(m)^{-1}u_i} R_{v_i} f(m) = L_{\check{u}_i} R_{v_i} f(m) = \tau(u_i) R_{v_i} f(m)$. Thus, we obtain the desired expression with $u_+ = \sum_{i=1}^{n} \varphi_i \otimes \tau(u_i) \otimes v_i$. $\square$

Let $U \subset M_{Q\sigma}$ be an open subset. It will be convenient to be able to refer to a 'formal application' of elements of the space $\mathcal{D}_{1Q}$, defined in (4.7), to $\mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$, the space of (formal) $\Delta_r(Q)$-exponential polynomial series with coefficients in $C^\infty(U, V_\tau)$, see the definition preceding Lemma 1.9. There is a natural way to define a formal application that is compatible with the expansions of Corollary 4.9 and with the map $u \mapsto u_*$, defined in the text following (4.7). The motivation for the following somewhat tedious chain of definitions will become clear in Lemma 4.11.

The product decomposition $M_{1Q} \simeq M_{Q\sigma} \times A_{Q\mathrm{q}}$ induces a natural isomorphism from $U(\mathfrak{m}_{1Q})$ onto $U(\mathfrak{m}_{Q\sigma}) \otimes U(\mathfrak{a}_{Q\mathrm{q}})$, by which we shall identify. Accordingly we have a natural isomorphism

$$\mathcal{D}_{1Q} \simeq {}^\circ\mathcal{D}_{1Q} \otimes U(\mathfrak{a}_{Q\mathrm{q}}), \tag{4.10}$$

where ${}^\circ\mathcal{D}_{1Q} := \mathcal{R}_Q \otimes \mathrm{End}(V_\tau) \otimes U(\mathfrak{m}_{Q\sigma})$. To each element $\varphi \in \mathcal{R}_Q$ we may associate its $\Delta_r(Q)$-exponential polynomial series of the form (4.6); this induces a linear embedding $\mathcal{R}_Q \to \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(M_{Q\sigma}))$ which by identity on the other tensor components may be extended to a linear embedding

$${}^\circ\mathcal{D}_{1Q} \to \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, \mathcal{D}_{Q\sigma}),$$

where $\mathcal{D}_{Q\sigma} := C^\infty(M_{Q\sigma}) \otimes \mathrm{End}(V_\tau) \otimes U(\mathfrak{m}_{Q\sigma})$. By identity on the second tensor component in (4.10) this embedding extends to a linear embedding

$$\mathrm{ep}\colon\ \mathcal{D}_{1Q} \to \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, \mathcal{D}_{Q\sigma}) \otimes U(\mathfrak{a}_{Q\mathrm{q}}). \tag{4.11}$$

The image $\mathrm{ep}(u)$ of an element $u \in \mathcal{D}_{1Q}$ under this embedding will be called the $\Delta_r(Q)$-exponential polynomial expansion of u. Via the right regular action of $U(\mathfrak{m}_{Q\sigma})$ we may naturally identify $\mathcal{D}_{Q\sigma}$ with the space of C^∞-differential operators acting on $C^\infty(M_{Q\sigma}, V_\tau)$. Accordingly, we have a continuous bilinear pairing $\mathcal{D}_{Q\sigma} \times C^\infty(U, V_\tau) \to C^\infty(U, V_\tau)$. This induces a formal application map from $\mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, \mathcal{D}_{Q\sigma}) \otimes \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$ to $\mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$ in the fashion described above Lemma 1.10. The image of an element of the form $u \otimes f$ under this map will be denoted by uf.

On the other hand, in Lemma 1.9 we described the formal application map $U(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau)) \to \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$. The image of an element of the form $v \otimes f$ under this map is denoted by vf. Combination of the above formal application maps leads to the formal application map

$$[\mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, \mathcal{D}_{Q\sigma}) \otimes U(\mathfrak{a}_{Q\mathrm{q}})] \otimes \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau)) \to \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau)),$$

given by $(u \otimes v) \otimes f \mapsto (u \otimes v)f := u(vf)$, for $u \in \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, \mathcal{D}_{Q\sigma})$, $v \in U(\mathfrak{a}_{Q\mathrm{q}})$ and $f \in \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$. Composing with the embedding (4.11) we finally obtain the linear map

$$\mathcal{D}_{1Q} \otimes \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau)) \to \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$$

given by $u \otimes f \mapsto uf := \mathrm{ep}(u)f$, for $u \in \mathcal{D}_{1Q}$ and $f \in \mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$. We shall call this map the formal application of $\mathcal{D}_{1Q}$ to $\mathcal{F}^{\mathrm{ep}}(A_{Q\mathrm{q}}, C^\infty(U, V_\tau))$.

Let now $R \geq 1$ and let $U \subset M_{Q\sigma}[R]$ be an open subset. We use the obvious natural isomorphism to identify the space $C^{\rm ep}(A^+_{Q{\rm q}}(R^{-1}), C^\infty(U, V_\tau))$ with a subspace of $C^\infty(UA^+_{Q{\rm q}}(R^{-1}), V_\tau)$. If $u \in \mathcal{D}_{1Q}$, then the associated differential operator u_* induces a map from the first space into the latter.

Lemma 4.11 *Let $u \in \mathcal{D}_{1Q}$, let $R \geq 1$ and let $U \subset M_{Q\sigma}[R]$ be an open subset. Then u_* maps the space $C^{\rm ep}(A^+_{Q{\rm q}}(R^{-1}), C^\infty(U, V_\tau))$ into itself. Moreover, if f belongs to that space, then the $\Delta_r(Q)$-exponential polynomial expansion of $u_* f$ is obtained from the formal application of u to the exponential polynomial expansion of f.*

Proof: This follows from retracing the definitions of u_* and of the formal application of u given above and applying Corollary 4.9 and Lemmas 1.9 and 1.10. $\square$

Given $v \in N_K(\mathfrak{a}_{\rm q})$ we define $\mu_{Q,v}: \mathbb{D}({\rm X}) \to \mathbb{D}({\rm X}_{1Q,v}) = \mathbb{D}(M_{1Q}/M_{1Q} \cap vHv^{-1})$ by

$$\mu_{Q,v} = {\rm Ad}(v) \circ \mu_{v^{-1}Qv},$$

where ${\rm Ad}(v): \mathbb{D}({\rm X}_{1v^{-1}Qv,e}) \to \mathbb{D}({\rm X}_{1Q,v})$ is induced by the restriction to $U(\mathfrak{m}_{1v^{-1}Qv})$ of ${\rm Ad}(v)$ on $U(\mathfrak{g})$. Then $\mu_{Q,v}$ depends on Q only through M_{1Q}. It is easily seen that

$$\mu_{Q,v} = \mu^v_Q \circ {\rm Ad}(v) \tag{4.12}$$

where $\mu^v_Q: \mathbb{D}({\rm X}_v) = \mathbb{D}(G/vHv^{-1}) \to \mathbb{D}({\rm X}_{1Q,v}) = \mathbb{D}(M_{1Q}/M_{1Q} \cap vHv^{-1})$ is defined similarly as μ_Q, but with H replaced by vHv^{-1}, and where ${\rm Ad}(v): \mathbb{D}({\rm X}) \to \mathbb{D}({\rm X}_v)$ is induced by ${\rm Ad}(v)$ on $U(\mathfrak{g})$.

Let $M_{Q\sigma,+} = M_{Q\sigma} \cap M_{1Q,+}$ and, for $R \geq 1$, $M_{Q\sigma,+}[R] = M_{Q\sigma}[R] \cap M_{1Q,+}$.

Lemma 4.12 *Let $f \in C^{\rm ep}({\rm X}_+ : \tau)$ and let $D \in \mathbb{D}({\rm X})$. Then $Df \in C^{\rm ep}({\rm X}_+ : \tau)$.*

Let $Q \in \mathcal{P}_\sigma$ and let $u_+ \in \mathcal{R}^+_Q \otimes {\rm End}(V_\tau) \otimes U(\mathfrak{m}_{1Q})$ be associated with D as in Proposition 4.10. Then the following holds.

(a) *The $\Delta_r(Q)$-exponential expansion of Df along (Q, e) is obtained by the formal application of $\mu'_Q(D) + u_+$ to the $\Delta_r(Q)$-exponential polynomial expansion of f along (Q, e).*

(b) *Let $v \in N_K(\mathfrak{a}_{\rm q})$, then ${\rm Exp}(Q, v \,|\, Df) \subset {\rm Exp}(Q, v \,|\, f) - \mathbb{N}\Delta_r(Q)$.*

(c) *If ξ is a leading exponent of f along (Q, v), then*

$$a^{\xi + \rho_Q} q_\xi(Q, v \,|\, Df, \log a, m) = [\mu_{Q,v}(D)\varphi](ma), \qquad (m \in M_{Q\sigma,+},\ a \in A_{Q{\rm q}}), \tag{4.13}$$

where the function $\varphi: M_{1Q,+} \to V_\tau$ is defined by $\varphi(ma) = a^{\xi+\rho_Q} q_\xi(Q, v \,|\, f, \log a, m)$, for $m \in M_{Q\sigma,+}$ and $a \in A_{Q{\rm q}}$.

Proof: Let $R \geq 1$ and let $\underline{f}$ be the function $A^+_{Q\mathrm{q}}(R^{-1}) \to C^\infty(M_{Q\sigma,+}[R], V_\tau)$ defined by $\underline{f}(a,m) = f(ma)$. It follows from the hypothesis on f and Theorem 3.4 that $\underline{f}(a,m)$ belongs to $C^{\mathrm{ep}}(A^+_{Q\mathrm{q}}(R^{-1}), C^\infty(M_{Q\sigma,+}[R], V_\tau))$. Moreover, its $\Delta_r(Q)$-exponential polynomial expansion coincides with the expansion of f along (Q,e). Put $u = \mu'_Q(D) + u_+$. Then it follows from the previous lemma that $u_*\underline{f}$ belongs to $C^{\mathrm{ep}}(A^+_{Q\mathrm{q}}(R^{-1}), C^\infty(M_{Q\sigma,+}[R], V_\tau))$; its expansion is obtained from the formal application of u to the (Q,e)-expansion of f. It follows from Theorem 3.4 that the expansion is independent of R and that its coefficients are functions in $C^\infty(M_{Q\sigma,+}, V_\tau)$. On the other hand, it follows from Proposition 4.10 that $u_*\underline{f}(a,m) = Df(ma)$. This implies that Df has a $\Delta_r(Q)$-exponential polynomial expansion along (Q,e) with coefficients in $C^\infty(M_{Q\sigma,+}, V_\tau)$. Since Df is right H-invariant, the coefficients are actually functions in $C^\infty(\mathrm{X}_{Q,e,+}, V_\tau)$. Moreover, the expansion is independent of R and converges neatly on $A^+_{Q\mathrm{q}}(R^{-1})$ as an expansion with coefficients in $C^\infty(\mathrm{X}_{Q,e,+}[R], V_\tau)$. In particular this holds for every minimal parabolic subgroup Q; hence $Df \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$.

In the above we have established assertion (a). It follows from this assertion that (b) holds with $v = 1$ for every $Q \in \mathcal{P}_\sigma$. By Lemma 3.6 it also holds for arbitrary $Q \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$.

It remains to establish (c). Assume first that $v = e$. Fix $\xi \in \mathrm{Exp}_\mathrm{L}(Q, e \,|\, f)$. Then by (a), $a^\xi q_\xi(Q, e \,|\, Df, \log a, m)$ is the term with exponent ξ in the series that arises from the formal application of $\mu'_Q(D) + u_+$ to the (Q,e)-expansion of f. The exponents of the expansion $\mathrm{ep}(u_+)$ of u_+ all belong to $-[\mathbb{N}\Delta_r(Q)] \setminus \{0\}$. The application of u_+ therefore gives rise to an expansion with exponents in $\mathrm{Exp}(Q, e \,|\, f) - [\mathbb{N}\Delta_r(Q)] \setminus \{0\}$. The latter set does not contain ξ, since ξ is leading. Hence $a^\xi q_\xi(Q, e \,|\, Df, \log a, m)$ is the term with exponent ξ in the expansion that arises from the formal application of $\mu'_Q(D)$ to the (Q,e)-expansion of f. Now $\mu'_Q(D) \in U(\mathfrak{m}_{1Q}) \simeq U(\mathfrak{m}_{Q\sigma}) \otimes U(\mathfrak{a}_{Q\mathrm{q}})$ and we see that the formal application of $\mu'_Q(D)$ to the (Q,e) expansion of f is induced by term by term differentiation in the $A_{Q\mathrm{q}}$ and the $M_{Q\sigma}$ variables. This implies that $a^\xi q_\xi(Q, e \,|\, Df, \log a, m) = [\mu'_Q(D)\varphi'](ma)$, where $\varphi'(ma) = a^\xi q_\xi(Q, e \,|\, f, \log a, m)$. This implies (4.13) for $v = e$.

Let now $v \in N_K(\mathfrak{a}_\mathrm{q})$ be arbitrary, and put $f^v = R_v f$. We shall apply the version of (4.13) just established to the expansion along (Q,e) of the function f^v on X_v. Let ξ be a leading exponent of f along (Q,v), then it follows from Lemma 3.7 that ξ is also a leading exponent of f^v along (Q,e). Moreover, let $D \in \mathbb{D}(\mathrm{X})$, then $(Df)^v = D^v f^v$ where $D^v := \mathrm{Ad}(v)D \in \mathbb{D}(\mathrm{X}_v)$. Hence

$$a^{\xi+\rho_Q} q_\xi(Q, e \,|\, (Df)^v, \log a, m) = [\mu^v_Q(D^v)\varphi](ma), \tag{4.14}$$

for $m \in M_{Q\sigma,+}$, $a \in A_{Q\mathrm{q}}$, where $\varphi(ma) = a^{\xi+\rho_Q} q_\xi(Q, e \,|\, f^v, \log a, m)$. It follows from Lemma 3.7 that $\varphi(ma) = a^{\xi+\rho_Q} q_\xi(Q, v \,|\, f, \log a, m)$, and $q_\xi(Q, e \,|\, (Df)^v) = q_\xi(Q, v \,|\, Df)$. Now (4.13) follows from (4.14) and (4.12). $\square$

Lemma 4.13 *Let $P \in \mathcal{P}^{\min}_\sigma$ and assume that $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$. Let $S \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ be a finite set as in Lemma 2.2, and let $D \in \mathbb{D}(\mathrm{X})$. Then* $\mathrm{Exp}(P, v \,|\, Df) \subset S - \mathbb{N}\Delta$

for every $v \in N_K(\mathfrak{a}_q)$ and, with notation as in Lemma 2.2,

$$(Df)_s = D(f_s). \tag{4.15}$$

Proof: It follows immediately from Lemma 4.12(b) that $\mathrm{Exp}(P, v \,|\, Df) \subset S - \mathbb{N}\Delta$ and that $\mathrm{Exp}(P, v \,|\, D(f_s)) \subset s - \mathbb{N}\Delta$ for $s \in S$. Now (4.15) follows from Lemma 2.2. $\square$

5 Spherical eigenfunctions

In this section we assume that (τ, V_τ) is a finite dimensional continuous representation of K. Let I be a cofinite ideal of the algebra $\mathbb{D}(\mathrm{X})$. Then by $C^\infty(\mathrm{X}_+ : \tau : I)$ we denote the space of $f \in C^\infty(\mathrm{X}_+ : \tau)$ satisfying the system of differential equations

$$Df = 0, \qquad (D \in I).$$

Remark 5.1 Many results of [3] that are formulated for $\mathbb{D}(\mathrm{X})$-finite τ-spherical functions on X are actually valid for the bigger class of $\mathbb{D}(\mathrm{X})$-finite functions in $C^\infty(\mathrm{X}_+ : \tau)$ as well, since their proofs only involve behavior of functions and operators on X_+. If such extended results are used in the text, we may give a reference to the present remark.

Remark 5.2 Let $v \in N_K(\mathfrak{a}_q)$. We recall from the text preceding Lemma 3.7 that right translation by v induces a topological linear isomorphism R_v from $C^\infty(\mathrm{X}_+ : \tau)$ onto the space $C^\infty(\mathrm{X}_{v,+} : \tau)$. It maps the subspace of $\mathbb{D}(\mathrm{X})$-finite functions onto the subspace of $\mathbb{D}(\mathrm{X}_v)$-finite functions. Thus, if $f \in C^\infty(\mathrm{X}_+ : \tau)$ is a $\mathbb{D}(\mathrm{X})$-finite function, then the theory of [3] may be applied to the $\mathbb{D}(\mathrm{X}_v)$-finite function $R_v f$; the results are then easily reformulated in terms of the function f.

Lemma 5.3 *Let $I \subset \mathbb{D}(\mathrm{X})$ be a cofinite ideal. Then $C^\infty(\mathrm{X}_+ : \tau : I) \subset C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$. In particular, the elements of $C^\infty(\mathrm{X}_+ : \tau : I)$ are real analytic functions on X_+. Moreover, there exists a finite set $X_I \subset \mathfrak{a}^*_{q\mathbb{C}}$ such that $\mathrm{Exp}_{\mathrm{L}}(P, v \,|\, f) \subset X_I$, for all $f \in C^\infty(\mathrm{X}_+ : \tau : I)$, $P \in \mathcal{P}_\sigma^{\min}$ and $v \in N_K(\mathfrak{a}_q)$.*

Proof: Let $Q \in \mathcal{P}_\sigma^{\min}$. Applying Theorem 2.5 of [3], see Remark 5.1, we obtain that $f|A_q^+(Q)$ is given by a neatly converging $\Delta(Q)$-exponential polynomial expansion for each $f \in C^\infty(\mathrm{X}_+ : \tau : I)$. Moreover, by Theorem 2.4 of [3], there exists a finite set $X_{I,Q,e} \subset \mathfrak{a}^*_{q\mathbb{C}}$, such that $\mathrm{Exp}_{\mathrm{L}}(f|A_q^+(Q)) \subset X_{I,Q,e}$. Let $w \in \mathcal{W}$. Applying the above argument to $R_w f$, cf. Remark 5.2, we see, more generally, that $T^{\downarrow}_{Q,w} f$ is given by the same type of expansion with leading exponents in a finite set $X_{I,Q,w} \subset \mathfrak{a}^*_{q\mathbb{C}}$ independent of f. This implies that $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$,

with $\mathrm{Exp}_{\mathrm{L}}(P, v \,|\, f) \subset X_I{:}= \cup_{Q,w} X_{I,Q,w}$, for all $P \in \mathcal{P}_\sigma^{\min}$ and $v \in \mathcal{W}$. Finally, if $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ is arbitrary, there exists $w \in \mathcal{W}$, $m \in K_{\mathrm{M}}$ and $h \in N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ such that $v = mwh$, and then $\mathrm{Exp}_{\mathrm{L}}(P, v \,|\, f) = \mathrm{Exp}_{\mathrm{L}}(P, w \,|\, f) \subset X_I$. □

Corollary 5.4 *Let $P \in \mathcal{P}_\sigma^{\min}$ and let $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ be a complete set of representatives of $W/W_{K\cap H}$. Let I be a cofinite ideal in $\mathbb{D}(\mathrm{X})$. Then there exists a finite set $S = S_I$ satisfying the properties of Lemma 2.2 for every $f \in C^\infty(\mathrm{X}_+ : \tau : I)$. Moreover, if S_I is any such set, then $f_s \in C^\infty(\mathrm{X}_+ : \tau : I)$ for every $f \in C^\infty(\mathrm{X}_+ : \tau : I)$ and all $s \in S_I$.*

Proof: This is an immediate consequence of Lemmas 5.3 and 4.13. □

The set X_I in Lemma 5.3 can be described more explicitly if the ideal I has codimension 1. Let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}_{\mathrm{q}}$, let $\Sigma(\mathfrak{b})$ be the restricted root system of $\mathfrak{b}$ in $\mathfrak{g}_{\mathbb{C}}$, and let $W(\mathfrak{b})$ be the associated reflection group.

Let γ be the Harish-Chandra isomorphism from $\mathbb{D}(\mathrm{X})$ onto the algebra $I(\mathfrak{b})$ of $W(\mathfrak{b})$-invariants in $S(\mathfrak{b})$, see [5], Sect. 2. To an element $\nu \in \mathfrak{b}_{\mathbb{C}}^*$ we associate the character $D \mapsto \gamma(D : \nu)$ of $\mathbb{D}(\mathrm{X})$ and denote its kernel by I_ν. Then I_ν is an ideal of codimension one in $\mathbb{D}(\mathrm{X})$; in fact, any codimension one ideal is of this form.

Let $W_0(\mathfrak{b})$ be the normalizer of $\mathfrak{a}_{\mathrm{q}}$ in $W(\mathfrak{b})$. Then restriction to $\mathfrak{a}_{\mathrm{q}}$ induces an epimorphism from $W_0(\mathfrak{b})$ onto W, cf. [5], Lemma 4.6. We put $\mathfrak{b}_{\mathrm{k}}{:}= \mathfrak{b} \cap \mathfrak{k}$. Then $\mathfrak{b} = \mathfrak{b}_{\mathrm{k}} \oplus \mathfrak{a}_{\mathrm{q}}$. Moreover, this decomposition is invariant under $W_0(\mathfrak{b})$.

Lemma 5.5 *There exists a finite subset $\mathcal{L} = \mathcal{L}_\tau$ of $\mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$ with the following property. Let $\nu \in \mathfrak{b}_{\mathbb{C}}^*$ and $f \in C^\infty(\mathrm{X}_+ : \tau : I_\nu)$. Let $P \in \mathcal{P}_\sigma^{\min}, v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and assume that $\xi \in \mathrm{Exp}_{\mathrm{L}}(P, v \,|\, f)$. Then*

$$\nu \in W(\mathfrak{b})(\mathcal{L} + \xi + \rho_P).$$

The proof is based on the following result, which will be proved first.

Lemma 5.6 *There exists a finite subset $\mathcal{L} = \mathcal{L}_\tau$ of $\mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$ with the following property. Let $\nu \in \mathfrak{b}_{\mathbb{C}}^*$ and $\varphi \in C^\infty(M_1/H_{\mathrm{M}_1} : \tau)$, and assume that*

$$\mu_P(D)\varphi = \gamma(D : \nu)\varphi$$

for all $D \in \mathbb{D}(\mathrm{X})$, where $\mu_P{:}\,\mathbb{D}(\mathrm{X}) \to \mathbb{D}(M_1/H_{\mathrm{M}_1})$ is as defined above Proposition 4.10, with $P \in \mathcal{P}_\sigma^{\min}$. Then $\varphi|_{A_{\mathrm{q}}}$ is a linear combination of exponential polynomials of the form $a \mapsto p(\log a)a^{w\nu}$, where $p \in P(\mathfrak{a}_{\mathrm{q}})$ and where $w \in W(\mathfrak{b})$ satisfies $w\nu|_{\mathfrak{b}_{\mathrm{k}}} \in \mathcal{L}$.

Proof: The algebra $\mathbb{D}(M/H_{\mathrm{M}})$ acts semisimply on $C^\infty(M/H_{\mathrm{M}} : \tau)$, see [5], Lemma 4.8; let $\mathcal{L}$ be the (finite) set of $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$ such that the associated character of $\mathbb{D}(M/H_{\mathrm{M}})$ occurs. We may assume that φ is a joint eigenfunction for

$\mathbb{D}(M/H_{\mathrm{M}})$, with eigenvalue character given by $\Lambda \in \mathcal{L}$. It follows that

$$(D\varphi)|_{A_{\mathrm{q}}} = \gamma_{\mathrm{M}_1}(D : \Lambda + \cdot)(\varphi|_{A_{\mathrm{q}}})$$

for $D \in \mathbb{D}(M_1/H_{\mathrm{M}_1}) \simeq \mathbb{D}(M/H_{\mathrm{M}}) \otimes S(\mathfrak{a}_{\mathrm{q}})$. Here γ_{M_1} denotes the Harish-Chandra isomorphism from $\mathbb{D}(M_1/H_{\mathrm{M}_1})$ into $S(\mathfrak{b})$, defined as in [9], above eq. (2.11), and $\gamma_{\mathrm{M}_1}(D : \Lambda + \cdot) \in S(\mathfrak{a}_{\mathrm{q}})$ is considered as a differential operator on A_{q}. Combining this identity with the assumption on φ, the identity $\gamma_{\mathrm{M}_1} \circ \mu_P = \gamma$, and the surjectivity of $\gamma: \mathbb{D}(\mathrm{X}) \to S(\mathfrak{b})^{W(\mathfrak{b})}$, it follows that

$$u(\Lambda + \cdot)(\varphi|_{A_{\mathrm{q}}}) = u(\nu)\varphi|_{A_{\mathrm{q}}}$$

for all $u \in S(\mathfrak{b})^{W(\mathfrak{b})}$. Let $\tilde{\varphi} \in C^\infty(\mathfrak{b})$ be defined by $\tilde{\varphi}(X+Y) = e^{\Lambda(X)}\varphi(\exp Y)$ for $X \in \mathfrak{b}_{\mathrm{k}}$, $Y \in \mathfrak{a}_{\mathrm{q}}$, then $u\tilde{\varphi} = u(\nu)\tilde{\varphi}$. This implies that $\tilde{\varphi}$ is a linear combination of exponential polynomials of the form $p\, e^{w\nu}$, where $p \in P(\mathfrak{b})$ and $w \in W(\mathfrak{b})$, see [27], Thm. III.3.13. However, from the definition of $\tilde{\varphi}$ it is readily seen that w only contributes if $w\nu|_{\mathfrak{b}_{\mathrm{k}}} = \Lambda$. $\square$

Proof of Lemma 5.5: We define the τ_{M}-spherical function $\varphi: M_1/M_1 \cap vHv^{-1} \simeq M/M \cap vHv^{-1} \times A_{\mathrm{q}} \to V_\tau$ by

$$\varphi(ma) = a^{\rho_P + \xi} q_\xi(P, v \,|\, f)(\log a, m).$$

Then it follows from the equation $Df = \gamma(D : \nu)f$ and Lemma 4.12 (c) applied to $D - \gamma(D : \nu)$ in place of D, that

$$\mu_{P,v}(D)\varphi = \gamma(D : \nu)\varphi.$$

Since φ is τ-spherical and non-zero, its restriction to A_{q} does not vanish.

Let first $v = e$, and let $\mathcal{L}$ be as in Lemma 5.6. It then follows immediately from that lemma that there exists $w \in W(\mathfrak{b})$ such that $w\nu|_{\mathfrak{b}_{\mathrm{k}}} \in \mathcal{L}$ and $w\nu|_{\mathfrak{a}_{\mathrm{q}}} = \xi + \rho_P$.

For general $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ we also obtain the result from Lemma 5.6, by applying it to the function $\varphi^v := \rho_{\tau, v^{-1}}\varphi$. Indeed, it follows from the definition of $\mu_{P,v}$ that φ^v satisfies the assumption of the lemma. Hence there exists $w \in W(\mathfrak{b})$ such that $w\nu|_{\mathfrak{b}_{\mathrm{k}}} \in \mathcal{L}$ and $w\nu|_{\mathfrak{a}_{\mathrm{q}}} = v^{-1}(\xi + \rho_P)$. Let $v' \in W_0(\mathfrak{b})$ be such that $v'Y = vY$ for all $Y \in \mathfrak{a}_{\mathrm{q}}$, then $\nu \in (v'w)^{-1}(v'\mathcal{L} + \xi + \rho_P)$. $\square$

We will also need a result on leading coefficients along non-minimal parabolic subgroups.

Lemma 5.7 *Let $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ be a $\mathbb{D}(\mathrm{X})$-finite function. Let $Q \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and assume that $\xi \in \mathrm{Exp}_{\mathrm{L}}(Q, v \,|\, f)$. Then the function $\varphi: \mathrm{X}_{1Q,v,+} \to V_\tau$ defined by*

$$\varphi(ma) = a^{\xi + \rho_Q} q_\xi(Q, v \,|\, f, \log a, m) \qquad (m \in \mathrm{X}_{Q,v,+},\ a \in A_{Q\mathrm{q}}),$$

is $\mathbb{D}(\mathrm{X}_{1Q,v})$-finite.

Proof: Let I be the annihilator of f in the algebra $\mathbb{D}(\mathrm{X})$. Then it follows from Lemma 4.12 (c) that $\mu_{Q,v}(D)\varphi = 0$ for all $D \in I$. The algebra $\mathbb{D}(\mathrm{X}_{1Q,v})$ is a finite module over the image of the homorphism $\mu_{Q,v}$; see [5], p. 342, and apply conjugation by v. Hence $\mu_{Q,v}(I)$ generates a cofinite ideal in $\mathbb{D}(\mathrm{X}_{1Q,v})$. This establishes the result. $\square$

We end this section with a result that limits the asymptotic exponents occurring in discrete series representations to a countable set. Later we will apply this result to exclude the possibility of a 'continuum of discrete series' (see the proof of Lemma 9.13).

To formulate the result we need to define asymptotic exponents for a K-finite rather than a τ-spherical function. We denote by $\widehat{K}$ the collection of equivalence classes of irreducible continuous representations of K. If $\vartheta \subset \widehat{K}$ is a finite subset, then by $C^\infty(\mathrm{X}_+)_\vartheta$ we denote the space of smooth K-finite functions in $C^\infty(\mathrm{X}_+)$ all of whose K-types belong to ϑ. By $V_\vartheta := C(K)_\vartheta$ we denote the space of left K-finite continuous functions on K all of whose left K-types belong to ϑ. Moreover, by τ_ϑ we denote the restriction of the right regular representation to V_ϑ. If $f \in C^\infty(\mathrm{X}_+)_\vartheta$, then the function $\varsigma_\vartheta(f)\colon \mathrm{X} \to V_\vartheta$, defined by $\varsigma_\vartheta(f)(x)(k) = f(kx)$ for $x \in \mathrm{X}_+, k \in K$ belongs to $C^\infty(\mathrm{X}_+ : \tau_\vartheta)$. The map $\varsigma := \varsigma_\vartheta$ is a topological linear isomorphism from $C^\infty(\mathrm{X}_+)_\vartheta$ onto $C^\infty(\mathrm{X}_+ : \tau_\vartheta)$, intertwining the $\mathbb{D}(\mathrm{X})$-actions on these spaces. Moreover, ς maps the closed subspace $C^\infty(\mathrm{X})_\vartheta$ of globally defined smooth functions onto the similar subspace $C^\infty(\mathrm{X} : \tau_\vartheta)$. We denote by $C^{\mathrm{ep}}(\mathrm{X}_+)_\vartheta$ the preimage of $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau_\vartheta)$ under ς. It follows from Lemma 5.3 that $\mathbb{D}(\mathrm{X})$-finite funcitons in $C^\infty(\mathrm{X}_+)_\vartheta$ belong to $C^{\mathrm{ep}}(\mathrm{X}_+)_\vartheta$. Let $f \in C^{\mathrm{ep}}(\mathrm{X}_+)_\vartheta$; then for $P \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ we define the set of exponents of f along (P, v) by

$$\mathrm{Exp}(P, v \,|\, f) := \mathrm{Exp}(P, v \,|\, \varsigma(f)).$$

Note that this collection is the union for $k \in K$ and $m \in \mathrm{X}_{P,v,+}$ of the collections of exponents occurring in the $\Delta(P)$-exponential polynomial expansions of $a \mapsto f(kamv)$.

Let $\mathcal{C}(\mathrm{X})$ denote the space of Schwartz functions on X, see [9], Section 6, and let $\mathcal{A}_2(\mathrm{X})_K$ denote the space of K-finite and $\mathbb{D}(\mathrm{X})$-finite functions $f \in \mathcal{C}(\mathrm{X})$. These functions are real analytic and belong to $L^2(\mathrm{X})$, cf. [3], Thm. 7.3.

Lemma 5.8 *Assume that the center of G is compact. Then*

$$\{\xi \in \mathrm{Exp}(P, v \,|\, f) \mid P \in \mathcal{P}_\sigma^{\min}, v \in N_K(\mathfrak{a}_{\mathrm{q}}), f \in \mathcal{A}_2(\mathrm{X})_K\}$$

is a countable subset of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

Proof: Let $\widehat{\mathrm{X}}_d$ denote the set of equivalence classes of discrete series representations of the symmetric space X. This set is countable, since $L^2(\mathrm{X})$ is a separable Hilbert space. Given $\omega \in \widehat{\mathrm{X}}_d$ we denote by $L^2(\mathrm{X})_\omega$ the collection of functions $f \in L^2(\mathrm{X})$ whose closed G-span in $L^2(\mathrm{X})$ is equivalent to a finite direct sum of copies of ω. Let $\widehat{K}$ denote the countable set of equivalence classes of irreducible

representations of K. Given $\omega \in \widehat{\mathrm{X}}_d$ and $\delta \in \widehat{K}$, we denote by $L^2(\mathrm{X})_{\omega,\delta}$ the collection of K-finite elements of type δ in $L^2(\mathrm{X})_\omega$. It follows from [3], Thm. 7.3, that $L^2(\mathrm{X})_{\omega,\delta}$ is a subspace of $\mathcal{A}_2(\mathrm{X})_K$, and from [2], Lemma 3.9, that this subspace is finite dimensional. On the other hand, let $f \in \mathcal{A}_2(\mathrm{X})_K$, and let $V \subset L^2(\mathrm{X})$ denote the closed G-span of f. It follows from [2], Lemma 3.9, that V is admissible. Since V is finitely generated, it must then be a finite direct sum of irreducible representations. This implies that f belongs to a finite direct sum of spaces $L^2(\mathrm{X})_{\omega,\delta}$. From the above we conclude that $\mathcal{A}_2(\mathrm{X})_K$ equals the following countable algebraic direct sum:

$$\mathcal{A}_2(\mathrm{X})_K = \oplus_{\omega\in\widehat{\mathrm{X}}_d,\, \delta\in\widehat{K}} \quad L^2(\mathrm{X})_{\omega,\delta}. \tag{5.1}$$

Let $\omega \in \widehat{\mathrm{X}}_d$ and $\delta \in \widehat{K}$. Then it follows from Lemma 5.3 and the finite dimensionality of $L^2(\mathrm{X})_{\omega,\delta}$ that there exists a countable subset $\mathcal{E}_{\omega,\delta} \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ such that

$$\mathrm{Exp}(P, v \,|\, f) \subset \mathcal{E}_{\omega,\delta}$$

for all $f \in L^2(\mathrm{X})_{\omega,\delta}, P \in \mathcal{P}_\sigma^{\min}, v \in N_K(\mathfrak{a}_\mathrm{q})$. Combining this observation with (5.1), we obtain the desired result. $\square$

6 Separation of exponents

Let $Q \in \mathcal{P}_\sigma$. In the next section we shall consider functions $f_\lambda \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, with parameter $\lambda \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, whose exponents along $P \in \mathcal{P}_\sigma^{\min}$ lie in sets of the form $W\lambda + S - \mathbb{N}\Delta(P)$, where $S \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ is a finite set. In general, given $\xi \in W\lambda + S - \mathbb{N}\Delta(P)$, the elements $s \in W/W_Q$ and $\eta \in S - \mathbb{N}\Delta(P)$, such that $\xi = s\lambda + \eta$, are not unique. In the present section we define a condition on λ that allows this unique determination for all ξ. In particular, the condition is valid for generic $\lambda \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. We consider also the case where P is non-minimal.

Let $P, Q \in \mathcal{P}_\sigma$. We define the equivalence relation $\sim_{P|Q}$ on W by

$$s \sim_{P|Q} t \quad \Longleftrightarrow \quad \forall \lambda \in \mathfrak{a}^*_{Q\mathrm{q}}\colon\ s\lambda|_{\mathfrak{a}_{P\mathrm{q}}} = t\lambda|_{\mathfrak{a}_{P\mathrm{q}}}. \tag{6.1}$$

The associated quotient is denoted by $W/\sim_{P|Q}$. We note that the classes in $W/\sim_{P|Q}$ are left W_P- and right W_Q-invariant. Thus, $W/\sim_{P|Q}$ may also be viewed as a quotient of $W_P\backslash W/W_Q$.

If $s, t \in W$ then one readily sees that $s \sim_{P|Q} t \iff s^{-1} \sim_{Q|P} t^{-1}$. Hence the anti-automorphism $s \mapsto s^{-1}$ of W factors to a bijection from $W/{\sim_{P|Q}}$ onto $W/{\sim_{Q|P}}$, which we denote by $\sigma \mapsto \sigma^{-1}$.

If $s \in W$ and $\lambda \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then the restriction $s\lambda|_{\mathfrak{a}_{P\mathrm{q}}}$ depends on s through its class $[s]$ in $W/{\sim_{P|Q}}$. We therefore agree to write

$$[s]\lambda|_{\mathfrak{a}_{P\mathrm{q}}} := s\lambda|_{\mathfrak{a}_{P\mathrm{q}}}.$$

Definition 6.1 *For $S \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ a finite subset, we define $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P,S)$ to be the subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ consisting of elements λ such that, for all $s_1, s_2 \in W$,*

$$(s_1\lambda - s_2\lambda)|_{\mathfrak{a}_{P\mathrm{q}}} \in [S + (-S)]|_{\mathfrak{a}_{P\mathrm{q}}} + \mathbb{Z}\Delta_r(P) \quad \Rightarrow \quad s_1 \sim_{P|Q} s_2.$$

Lemma 6.2 *Let $S \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ be finite. Then, for $\lambda \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$,*

$$W\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + (S - \mathbb{N}\Delta(P))|_{\mathfrak{a}_{P\mathrm{q}}} = \bigcup_{\sigma \in W/\sim_{P|Q}} \left(\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + (S - \mathbb{N}\Delta(P))|_{\mathfrak{a}_{P\mathrm{q}}}\right).$$

Moreover, the union is disjoint if and only if $\lambda \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P,S)$.

Proof: Straightforward. □

Lemma 6.3 *Let $Q, P \in \mathcal{P}_\sigma$, and let S be a finite subset of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Then $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P,S)$ equals the complement of the union of a locally finite collection of proper affine subspaces in $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$.*

Proof: Let $p\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$ denote the map induced by restriction to $\mathfrak{a}_{P\mathrm{q}}$. Let Π be the complement of the diagonal in the set $W/\sim_{P|Q} \times W/\sim_{P|Q}$. Then for every $\sigma = (\sigma_1, \sigma_2) \in \Pi$ and every $\eta \in \mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$ we write $\mathcal{A}_{\sigma,\eta} = \{\lambda \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \mid p(\sigma_1\lambda - \sigma_2\lambda) = \eta\}$. Note that $\mathcal{A}_{\sigma,0}$ is a proper affine subspace of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$. If $\lambda \in \mathcal{A}_{\sigma,\eta}$ then $\mathcal{A}_{\sigma,\eta}$ equals $\lambda + \mathcal{A}_{\sigma,0}$; hence the set $\mathcal{A}_{\sigma,\eta}$ is either empty or a proper affine subspace.

Let $\mathcal{A}$ be the collection of subsets of the form $\mathcal{A}_{\sigma,\xi}$, for $\sigma \in \Pi$ and $\xi \in p(S + (-S)) + \mathbb{Z}\Delta_r(P)$. Then $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P,S)$ equals the complement of $\cup\mathcal{A}$ in $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$. Thus, it remains to show that the collection $\mathcal{A}$ is locally finite.

Let $\mathcal{C}$ be a compact subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and let X be the collection of $\xi \in p(S + (-S)) + \mathbb{Z}\Delta_r(P)$ such that $\mathcal{C} \cap \mathcal{A}_{\sigma,\xi} \neq \emptyset$ for some $\sigma \in \Pi$. Then it suffices to show that X is finite.

Let $\mathcal{C}' \subset \mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$ be the image of $\Pi \times \mathcal{C}$ under the map $(\sigma, \lambda) \mapsto p(\sigma_1\lambda - \sigma_2\lambda)$. Then X equals the intersection of $\mathcal{C}'$ with $p(S + (-S)) + \mathbb{Z}\Delta_r(P)$. The latter set is discrete since S is finite, whereas the elements of $\Delta_r(P)$ are linearly independent. It follows that X is finite. □

Remark 6.4 In particular, it follows from the above lemma that $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P,S)$ is a full open subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$; see Appendix B for the notion of full.

Lemma 6.5 *Let $Q, P \in \mathcal{P}_\sigma$. If either $\mathfrak{a}_{Q\mathrm{q}}$ or $\mathfrak{a}_{P\mathrm{q}}$ has codimension at most 1 in $\mathfrak{a}_{\mathrm{q}}$, then the natural projection $W_P\backslash W/W_Q \to W/\sim_{P|Q}$ is a bijection.*

Proof: It suffices to prove injectivity of the map. Since $s \mapsto s^{-1}$ induces a bijection from $W/\sim_{P|Q}$ onto $W/\sim_{Q|P}$, it suffices to prove this when $\mathfrak{a}_{P\mathrm{q}}$ has codimension at most 1. We assume the latter to hold.

For $s \in W$, let $[s]$ denote its canonical image in $W/\sim_{P|Q}$. Assume that $s, t \in W$ and that $[s] = [t]$. Then for every $\lambda \in \mathfrak{a}_{Q\mathrm{q}}^*$ we have $s\lambda = t\lambda$ on $\mathfrak{a}_{P\mathrm{q}}$. If $\mathfrak{a}_{P\mathrm{q}} = \mathfrak{a}_\mathrm{q}$, this implies that $s = t$ on $\mathfrak{a}_{Q\mathrm{q}}^*$, hence $sW_Q = tW_Q$, and since W_P is trivial in this case, the proof is finished. Thus, we may as well assume that $\mathfrak{a}_{P\mathrm{q}}$ has codimension 1. Then there exists a root $\alpha \in \Sigma$ such that $\mathfrak{a}_{P\mathrm{q}} = \ker \alpha$. Note that $W_P = \{1, s_\alpha\}$. For every $\lambda \in \mathfrak{a}_{Q\mathrm{q}}^*$ the Weyl group images $s\lambda$ and $t\lambda$ have equal length in $\mathfrak{a}_\mathrm{q}^*$ and equal image under the orthogonal projection to $\alpha^\perp$. Hence there exists a constant $\eta \in \{0, 1\}$ such that $\langle s\lambda\,,\, \alpha\rangle = (-1)^\eta \langle t\lambda\,,\,\alpha\rangle$ for all $\lambda \in \mathfrak{a}_{Q\mathrm{q}}^*$. It follows that $s\lambda = s_\alpha^\eta t\lambda$ for all $\lambda \in \mathfrak{a}_{Q\mathrm{q}}^*$; hence $sW_Q = s_\alpha^\eta tW_Q$, from which it follows in turn that s and t have the same image in $W_P\backslash W/W_Q$. $\square$

In particular, if P is minimal, then the natural map $W/W_Q \to W/\sim_{P|Q}$ is a bijection; we shall use it to identify the sets involved.

7 Analytic families of spherical functions

In this section we assume that (τ, V_τ) is a finite dimensional continuous representation of K. Let $Q \in \mathcal{P}_\sigma$ and let Y be a finite subset of ${}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$, see (3.4).

In the following definition we introduce a space of analytic families of τ-spherical functions that will play a crucial role in the rest of this paper.

Definition 7.1 *Let Q, Y be as above and let $\Omega \subset \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ be an open subset. We define*

$$C_{Q,Y}^{\mathrm{ep}}(\mathrm{X}_+ : \tau : \Omega) \tag{7.1}$$

to be the space of C^∞-functions $f\colon \Omega \times \mathrm{X}_+ \to V_\tau$ satisfying the following conditions.

(a) *For every $\lambda \in \Omega$ the function $f_\lambda\colon x \mapsto f(\lambda, x)$ belongs to $C^\infty(\mathrm{X}_+ : \tau)$.*

(b) *There exists a constant $k \in \mathbb{N}$, and, for every $P \in \mathcal{P}_\sigma^{\min}$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$, a collection of functions $q_{s,\xi}(P, v\,|\,f) \in P_k(\mathfrak{a}_\mathrm{q}) \otimes \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{0,v} : \tau_\mathrm{M}))$, for $s \in W/W_Q$ and $\xi \in -sW_QY + \mathbb{N}\Delta(P)$, with the following property. For all $\lambda \in \Omega$, $m \in \mathrm{X}_{0,v}$ and $a \in A_\mathrm{q}^+(P)$,*

$$f_\lambda(mav) = \sum_{s \in W/W_Q} a^{s\lambda - \rho_P} \sum_{\xi \in -sW_QY + \mathbb{N}\Delta(P)} a^{-\xi} q_{s,\xi}(P, v\,|\,f, \log a)(\lambda, m), \tag{7.2}$$

where the $\Delta(P)$-exponential polynomial series with coefficients in V_τ is neatly convergent on $A_\mathrm{q}^+(P)$.

(c) *For every $P \in \mathcal{P}_\sigma^{\min}, v \in N_K(\mathfrak{a}_\mathrm{q})$ and $s \in W/W_Q$, the series*

$$\sum_{\xi \in -sW_QY + \mathbb{N}\Delta(P)} a^{-\xi} q_{s,\xi}(P, v\,|\,f, \log a)$$

converges neatly on $A_{\mathrm{q}}^{+}(P)$ as a $\Delta(P)$-exponential polynomial series with coefficients in $\mathcal{O}(\Omega, C^{\infty}(\mathrm{X}_{0,v} : \tau_{\mathrm{M}}))$.

If $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$, we define the asymptotic degree of f, denoted $\deg_{\mathrm{a}} f$, to be the smallest number $k \in \mathbb{N}$ for which the above condition (b) is fulfilled.

Remark 7.2 We note that the space (7.1) depends on Q through its σ-split component $A_{Q\mathrm{q}}$. Moreover, from Lemma 3.6 we see that in the above definition it suffices to require (b) and (c) for a fixed given $P \in \mathcal{P}_\sigma^{\min}$ and for each v in a given set $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ of representatives for $W/W_{K\cap H}$. Alternatively, by the same lemma it suffices to require (b) and (c) for a fixed given $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and arbitrary $P \in \mathcal{P}_\sigma^{\min}$.

Lemma 7.3 *Let $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. Then $f_\lambda \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ and*

$$\mathrm{Exp}(P, v \,|\, f_\lambda) \subset W(\lambda + Y) - \rho_P - \mathbb{N}\Delta(P) \tag{7.3}$$

for all $\lambda \in \Omega$, $P \in \mathcal{P}_\sigma^{\min}$, and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$. Moreover, let $\Omega' := \Omega \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY)$ (see Definition 6.1). Then Ω' is open dense in Ω and

$$q_{s,\xi}(P, v \,|\, f, X, \lambda) = q_{s\lambda - \rho_P - \xi}(P, v \,|\, f_\lambda, X) \tag{7.4}$$

for every $s \in W/W_Q$, $\xi \in -sW_Q Y + \mathbb{N}\Delta(P)$, $X \in \mathfrak{a}_{\mathrm{q}}$ and $\lambda \in \Omega'$. In particular, the functions $q_{s,\xi}(P, v \,|\, f)$ are uniquely determined.

Proof: The first statement and (7.3) follow immediately from condition (b) in the above definition. The set Ω' is open dense in Ω by Lemma 6.3, and it follows from Lemmas 6.2 and 6.5 that if $\lambda \in \Omega'$ then the sets $s(\lambda + W_Q Y) - \rho_P - \mathbb{N}\Delta(P)$, $s \in W/W_Q$, are mutually disjoint. Then (7.4) holds by uniqueness of asymptotics. □

The following result shows that an element of $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ may be viewed as an analytic family of spherical functions.

Lemma 7.4 *Let $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. Then $\lambda \mapsto f_\lambda$ is a holomorphic function on Ω with values in $C^{\infty}(\mathrm{X}_+ : \tau)$.*

Proof: Let $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ be a complete set of representatives for $W/W_{K\cap H}$. Note that for $v \in \mathcal{W}$ the $V_\tau^{K_{\mathrm{M}} \cap vHv^{-1}}$-valued function $T^{\downarrow}_{P,v} f_\lambda$ on $A_{\mathrm{q}}^{+}(P)$ is given by the series on the right-hand side of (7.2) with $m = e$. It follows from condition (c) of Definition 7.1 that $a \mapsto T^{\downarrow}_{P,v} f_\lambda(a)$ defines a smooth function on $A_{\mathrm{q}}^{+}(P)$ with values in $\mathcal{O}(\Omega) \otimes V_\tau^{K_{\mathrm{M}} \cap vHv^{-1}}$. According to Appendix A, the function $\lambda \mapsto T^{\downarrow}_{P,v} f_\lambda(\,\cdot\,)$ is a holomorphic function on Ω with values in $C^{\infty}(A_{\mathrm{q}}^{+}(P), V_\tau^{K_{\mathrm{M}} \cap vHv^{-1}})$. Hence $\lambda \mapsto T^{\downarrow}_{P,\mathcal{W}}(f_\lambda)$ is a holomorphic function on Ω with values in $C^{\infty}(A_{\mathrm{q}}^{+}(P), \oplus_{v\in\mathcal{W}} V_\tau^{K_{\mathrm{M}} \cap vHv^{-1}})$. The conclusion of the lemma now follows by application of the isomorphism (2.8). □

If Ω', Ω are open subsets of $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ with $\Omega' \subset \Omega$, then restriction from $\Omega \times \mathrm{X}_+$ to $\Omega' \times \mathrm{X}_+$ obviously induces a linear map

$$\rho^{\Omega}_{\Omega'}: C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega) \to C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega'). \tag{7.5}$$

Accordingly, the assignment

$$\Omega \mapsto C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega), \tag{7.6}$$

defines a presheaf of complex linear spaces on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Here we agree that (7.6) assigns the trivial space to the empty set.

The following lemma will be useful at a later stage.

Lemma 7.5 *Let $Q \in \mathcal{P}_\sigma$ and $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset.*

(a) *If Ω', Ω are open subsets of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ with $\Omega' \neq \emptyset$, Ω connected and $\Omega' \subset \Omega$, then the restriction map (7.5) is injective. Moreover, $\deg_{\mathrm{a}}(\rho^{\Omega}_{\Omega'} f) = \deg_{\mathrm{a}}(f)$ for all $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$.*

(b) *The presheaf (7.6) is a sheaf.*

Proof: The injectivity of the restriction map follows by analytic continuation, in view of Lemma 7.4. Let $f' = \rho^{\Omega}_{\Omega'} f$. Let $P \in \mathcal{P}^{\min}_\sigma$, $v \in N_K(\mathfrak{a}_{\mathrm{q}})$, $s \in W/W_Q$ and $\xi \in -sW_Q\lambda + \mathbb{N}\Delta(P)$ then it follows from (7.4) that

$$q_{s,\xi}(P, v \,|\, f', \,\cdot\,, \lambda) = q_{s,\xi}(P, v \,|\, f, \,\cdot\,, \lambda) \tag{7.7}$$

for λ in a dense open subset of Ω', hence for all $\lambda \in \Omega'$. In particular this implies that the polynomial degree of the function on the left-hand side of the equation is bounded by $\deg_{\mathrm{a}}(f)$; hence $\deg_{\mathrm{a}}(f') \leq \deg_{\mathrm{a}}(f)$. To prove the converse inequality, we note that the polynomial on the left-hand side of (7.7) is of degree at most $k' := \deg_{\mathrm{a}}(f')$ by the definition of the latter number. Since Ω is connected, it follows by analytic continuation that $\deg q_{s,\xi}(P, v \,|\, f, \,\cdot\,, \lambda) \leq k'$ for all $\lambda \in \Omega$. Since this holds for all P, v, σ, ξ, it follows that $\deg_{\mathrm{a}}(f) \leq k'$ and we obtain (a).

Assertion (b) is equivalent with the assertion that the presheaf satisfies the localization property (see [31], p. 9). This is established in a straightforward manner, by using (a). $\square$

We shall now discuss the action of invariant differential operators on families. If f is a family in $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$, and $D \in \mathbb{D}(\mathrm{X})$, then we define the family $Df: \Omega \times \mathrm{X}_+ \to V_\tau$ by

$$(Df)_\lambda = D(f_\lambda), \qquad (\lambda \in \Omega). \tag{7.8}$$

Proposition 7.6 *Let $f \in C^{\rm ep}_{Q,Y}({\rm X}_+ : \tau : \Omega)$. Then, for every $D \in \mathbb{D}({\rm X})$, the family Df belongs to $C^{\rm ep}_{Q,Y}({\rm X}_+ : \tau : \Omega)$; moreover, $\deg_{\rm a}(Df) \leq \deg_{\rm a}(f)$.*

Proof: Let $D \in \mathbb{D}({\rm X})$. Then $g = Df$ is a smooth function $\Omega \times {\rm X}_+ \to V_\tau$; moreover, for $\lambda \in \Omega$ the function $(Dg)_\lambda = Df_\lambda$ is τ-spherical. Thus, g satisfies condition (a) of Definition 7.1 and it remains to establish properties (b) and (c). In view of Remark 7.2 it suffices to do this for $v = e$ and arbitrary $P \in \mathcal{P}_\sigma^{\rm min}$. Let $k := \deg_{\rm a} f$.

It follows from condition (b) of Definition 7.1 that, for $\lambda \in \Omega$, the function f_λ belongs to $C^{\rm ep}({\rm X}_+ : \tau)$; moreover, its (P, e)-expansion is given by

$$f_\lambda(ma) = \sum_{s \in W/W_Q} a^{s\lambda - \rho_P} \sum_{\xi \in -sW_Q Y + \mathbb{N}\Delta(P)} a^{-\xi} q_{s,\xi}(P, e \,|\, f, \log a)(\lambda, m), \quad (7.9)$$

for $a \in A_{\rm q}^+(P)$ and $m \in M$. Let $u := \mu'_P(D) + u_+$ be the element of $\mathcal{D}_{1P}$ associated with D as in Proposition 4.10 with P in place of Q. In view of Corollary 4.9 its expansion ${\rm ep}(u)$, defined as in (4.11), is the sum, as i ranges over a finite index set I, of series of the form

$${\rm ep}(u)_i = \sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \varphi_{i,\nu} \otimes S_{i,\nu} \otimes u_{i,\nu} \otimes v_{i,\nu}.$$

Here $\varphi_{i,\nu} \in C^\infty(M_\sigma)$, $S_{i,\nu} \in {\rm End}(V_\tau)$, $u_{i,\nu} \in U(\mathfrak{m}_\sigma)$ and $v_{i,\nu} \in U(\mathfrak{a}_{\rm q})$, and $\deg(u_{i,\nu}) + \deg(v_{i,\nu}) \leq d := \deg(D)$ for all i, ν. By Lemma 4.12, the function g_λ belongs to $C^{\rm ep}({\rm X}_+ : \tau)$, for $\lambda \in \Omega$, and its (P, e) expansion results from (7.9) by the formal application of the element ${\rm ep}(u)$. This gives, for $\lambda \in \Omega$, $m \in M$ and $a \in A_{\rm q}^+(P)$, the neatly converging exponential polynomial expansion

$$g_\lambda(ma) = \sum_{s \in W/W_Q} a^{s\lambda - \rho_P} \sum_{\eta \in -sW_Q Y + \mathbb{N}\Delta(P)} a^{-\eta} \tilde{q}_{s,\eta}(\log a)(\lambda, m),$$

where $\tilde{q}_{s,\eta}$ is given by the following finite sum

$$\tilde{q}_{s,\eta}(X)(\lambda, m) := \sum_{i \in I} \sum_{\substack{\nu \in \mathbb{N}\Delta(P) \\ \xi \in -sW_Q Y + \mathbb{N}\Delta(P) \\ \nu + \xi = \eta}} \varphi_{i,\nu}(m) S_{i,\nu} [\, q_{s,\xi}(P, e \,|\, f, X; T_{s\lambda - \rho_P - \xi}(v_{i,\nu}), \lambda, m; u_{i,\nu}) \,],$$

for $\lambda \in \Omega$, $X \in \mathfrak{a}_{\rm q}$ and $m \in M$. Here we have used Harish-Chandra's convention to indicate by a semicolon on the left or right-hand side of a Lie group variable the differentiation on the corresponding side, with respect to that variable, by elements of the appropriate universal enveloping algebra. Moreover, given $\gamma \in \mathfrak{a}_{{\rm q}\mathbb{C}}^*$ we have denoted by T_γ the automorphism of $U(\mathfrak{a}_{\rm q})$ determined by $T_\gamma(X) = X + \gamma(X)$ for $X \in \mathfrak{a}_{\rm q}$.

From the above formula it readily follows that $\tilde{q}_{s,\eta}(X, \lambda)$ is a smooth function of (X, λ) with values in $C^\infty(M, V_\tau)$; moreover, it is polynomial in X of degree at most k and holomorphic in $\lambda \in \Omega$. This establishes condition (b) of Definition 7.1 with $v = e$, arbitrary $P \in \mathcal{P}_\sigma^{\rm min}$, and with

$$q_{s,\eta}(P, e \,|\, g) = \tilde{q}_{s,\eta}, \qquad (s \in W/W_Q,\ \eta \in -sW_Q Y + \mathbb{N}\Delta(P)).$$

For condition (c) we note that the series

$$\sum_{\eta \in -sW_Q Y + \mathbb{N}\Delta(P)} a^{-\eta} q_{s,\eta}(P, e \,|\, g, \log a) \tag{7.10}$$

arises from the series

$$\sum_{\xi \in -sW_Q Y + \mathbb{N}\Delta(P)} a^{-\xi} q_{s,\xi}(P, e \,|\, f, \log a) \tag{7.11}$$

by the formal application of $\mathrm{ep}(u)$ conjugated with multiplication by $a^{-s\lambda+\rho_P}$. From this we see that (7.10) arises from (7.11) by the formal application of the series

$$\sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \sum_{i \in I} \varphi_{i,\nu} \otimes S_{i,\nu} \otimes u_{i,\nu} \otimes v_{i,\nu}(\lambda),$$

with $v_{i,\nu}(\lambda) = T_{s\lambda-\rho_P}(v_{i,\nu})$. We now observe that $\lambda \mapsto T_{s\lambda-\rho_P}|_{U_d(\mathfrak{a}_{Q\mathrm{q}})}$ is a polynomial $\mathrm{End}(U_d(\mathfrak{a}_{Q\mathrm{q}}))$-valued function, of degree at most d. Hence there exists a finite set J and elements $p_j \in P_d(\mathfrak{a}^*_{Q\mathrm{q}})$ and $T_j \in \mathrm{End}(U_d(\mathfrak{a}_{Q\mathrm{q}}))$, for $j \in J$, such that

$$T_{s\lambda-\rho_P}|_{U_d(\mathfrak{a}_{Q\mathrm{q}})} = \sum_{j \in J} p_j(\lambda) T_j.$$

Let $B_{i,\nu,j}$ be the continuous endomorphism of $\mathcal{O}(\Omega, C^\infty(M_\sigma, V_\tau))$ defined by

$$B_{i,\nu,j}\psi(\lambda)(m) = p_j(\lambda)\varphi_{i,\nu}(m) S_{i,\nu}[\psi(\lambda)(m; u_{i,\nu})].$$

Then the series (7.10) arises from the formal application of the series

$$\sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \sum_{\substack{i \in I \\ j \in J}} B_{i,\nu,j} \otimes T_j(v_{i,\nu})$$

with coefficients in $\mathrm{End}(\mathcal{O}(\Omega, C^\infty(M_\sigma, V_\tau))) \otimes U(\mathfrak{a}_{Q\mathrm{q}})$ to (7.11), viewed as a series with coefficients in $\mathcal{O}(\Omega, C^\infty(M_\sigma, V_\tau))$. It follows from Lemmas 1.9 and 1.10 that the resulting series is neatly convergent as a series on $A^+_{\mathrm{q}}(P)$ with coefficients in $\mathcal{O}(\Omega, C^\infty(M_\sigma, V_\tau))$. This establishes (c) with $v = e$ and arbitrary $P \in \mathcal{P}^{\min}_\sigma$. $\square$

We will now describe the asymptotic behavior along walls for a family. If $P, Q \in \mathcal{P}_\sigma$ and $\sigma \in W/\!\sim_{P|Q}$ (see (6.1)), then for every subset $Y \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ we put

$$\sigma \cdot Y := \{ s\eta|_{\mathfrak{a}_{P\mathrm{q}}} \mid s \in W, [s] = \sigma,\ \eta \in Y \}. \tag{7.12}$$

Theorem 7.7 (Behavior along the walls). *Let $Q \in \mathcal{P}_\sigma$, $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset and $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset. Let $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ and let $k = \deg_{\mathrm{a}}(f)$.*

Let $P \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$. Then $\mathrm{Exp}(P, v \,|\, f_\lambda) \subset W(\lambda + Y)|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \mathbb{N}\Delta_r(P)$ *for every $\lambda \in \Omega$. Moreover, there exist unique functions*

$$q_{\sigma,\xi}(P, v \,|\, f) \in P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+} : \tau_P)),$$

for $\sigma \in W/\!\sim_{P|Q}$ and $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$, with the following property. For all $\lambda \in \Omega$, $m \in \mathrm{X}_{P,v,+}$ and $a \in A^+_{P\mathrm{q}}(R_{P,v}(m)^{-1})$,

$$f_\lambda(mav) = \sum_{\sigma \in W/\sim_{P|Q}} a^{\sigma\lambda - \rho_P} \sum_{\xi \in -\sigma\cdot Y + \mathbb{N}\Delta_r(P)} a^{-\xi}\, q_{\sigma,\xi}(P, v \,|\, f, \log a)(\lambda, m), \tag{7.13}$$

where the $\Delta_r(P)$-exponential polynomial series with coefficients in V_τ is neatly convergent on $A^+_{P\mathrm{q}}(R_{P,v}(m)^{-1})$. In particular, if $\lambda \in \Omega' := \Omega \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY)$ then

$$q_{\sigma,\xi}(P, v \,|\, f)(X, \lambda) = q_{\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi}(P, v \,|\, f_\lambda, X), \tag{7.14}$$

for $X \in \mathfrak{a}_{P\mathrm{q}}$.

Finally, for each $\sigma \in W/\!\sim_{P|Q}$ and every $R > 1$, the series

$$\sum_{\xi \in -\sigma\cdot Y + \mathbb{N}\Delta_r(P)} a^{-\xi} q_{\sigma,\xi}(P, v \,|\, f, \log a) \tag{7.15}$$

converges neatly on $A^+_{P\mathrm{q}}(R^{-1})$ as a $\Delta_r(P)$-exponential polynomial series with coefficients in $\mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+}[R] : \tau_P))$.

Proof: Let $P \in \mathcal{P}_\sigma$ and let $v \in N_K(\mathfrak{a}_\mathrm{q})$. Fix a minimal parabolic subgroup $P_1 \in \mathcal{P}^{\min}_\sigma$, contained in P. Fix a set $\mathcal{W}_{P,v} \subset N_{K_P}(\mathfrak{a}_\mathrm{q})$ of representatives for $W_P / W_P \cap v W_{K\cap H} v^{-1}$. Then the natural map $N_K(\mathfrak{a}_\mathrm{q}) \to W$ induces an embedding $\mathcal{W}_{P,v} v \hookrightarrow W/W_{K\cap H}$. Therefore, we may fix a set $\mathcal{W} \subset N_K(\mathfrak{a}_\mathrm{q})$ of representatives for $W/W_{K\cap H}$ containing $\mathcal{W}_{P,v} v$.

Fix $\lambda \in \Omega$ for the moment. Then by Lemma 7.3, the function f_λ belongs to $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, and $\mathrm{Exp}(P_1, w \,|\, f_\lambda) \subset W(\lambda + Y) - \rho_{P_1} - \mathbb{N}\Delta(P_1)$, for every $w \in N_K(\mathfrak{a}_\mathrm{q})$. According to Theorem 3.5, for every $u \in \mathcal{W}_{P,v}$, the set $\mathrm{Exp}(P, v \,|\, f_\lambda)_{P_1,u}$ is contained in $\mathrm{Exp}(P_1, uv \,|\, f_\lambda)|_{\mathfrak{a}_{P\mathrm{q}}}$. Hence, by of (3.20) with P and P_1 in place of Q and P, respectively, we infer that

$$\begin{aligned} \mathrm{Exp}(P, v \,|\, f_\lambda) &\subset [W(\lambda + Y) - \rho_{P_1} - \mathbb{N}\Delta(P_1)]|_{\mathfrak{a}_{P\mathrm{q}}} \\ &= W(\lambda + Y)|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \mathbb{N}\Delta_r(P). \end{aligned} \tag{7.16}$$

Notice that (7.14) is a consequence of (7.13), by Lemma 6.2. Therefore the functions $q_{\sigma,\xi}(P, v \,|\, f)$ are unique. We will now establish their existence.

It follows form (7.16) that the elements of $\mathrm{Exp}(P, v \,|\, f_\lambda)$ are all of the form $\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi$, with $\sigma \in W/\!\sim_{P|Q}$ and $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$. Fix such elements σ and ξ. Then by transitivity of asymptotics, cf. Theorem 3.5, we have, for

$u \in \mathcal{W}_{P,v}$, $X \in \mathfrak{a}_{P\mathrm{q}}$, $m \in M$ and $b \in {}^*A^+_{P\mathrm{q}}({}^*P_1)$, that

$$q_{\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}}-\rho_P-\xi}(P, v \,|\, f_\lambda, X, mbu) = \sum_{\substack{\zeta \in \mathrm{Exp}(P_1, uv \,|\, f_\lambda) \\ \zeta|_{\mathfrak{a}_{P\mathrm{q}}} = \sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi}} b^\zeta q_\zeta(P_1, uv \,|\, f_\lambda, X + \log b, m), \tag{7.17}$$

where the $\Delta_P(P_1)$-exponential polynomial series in the variable b converges neatly on ${}^*A^+_{P\mathrm{q}}({}^*P_1)$. It follows from condition (b) in Definition 7.1 that, for $\zeta \in \mathrm{Exp}(P_1, uv \,|\, f_\lambda)$,

$$q_\zeta(P_1, uv \,|\, f_\lambda, X + \log b, m) = \sum_{\substack{s \in W/W_Q \\ \mu \in -sW_Q Y + \mathbb{N}\Delta(P_1) \\ s\lambda - \rho_{P_1} - \mu = \zeta}} q_{s,\mu}(P_1, uv \,|\, f, X + \log b)(\lambda, m). \tag{7.18}$$

Now assume that λ is contained in the full (cf. Lemma 6.3) subset Ω' of Ω. Then, if $s \in W$ and $\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1)$ satisfy $[s\lambda - \rho_{P_1} - \mu]|_{\mathfrak{a}_{P\mathrm{q}}} = \sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi$, it follows that $[s] = \sigma$ and $\mu|_{\mathfrak{a}_{P\mathrm{q}}} = \xi$, see Lemma 6.2. Hence, combining (7.17) and (7.18) we infer that for $\lambda \in \Omega'$, $u \in \mathcal{W}_{P,v}$, $X \in \mathfrak{a}_{P\mathrm{q}}$, $m \in M$ and $b \in {}^*A^+_{P\mathrm{q}}({}^*P_1)$,

$$\begin{aligned} &q_{\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}}-\rho_P-\xi}(P, v \,|\, f_\lambda, X, mbu) = \\ &\quad = \sum_{\substack{s \in W/W_Q \\ [s] = \sigma}} b^{s\lambda - \rho_{P_1}} \sum_{\substack{\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1) \\ \mu|_{\mathfrak{a}_{P\mathrm{q}}} = \xi}} b^{-\mu} q_{s,\mu}(P_1, uv \,|\, f, X + \log b, \lambda)(m). \end{aligned} \tag{7.19}$$

It will be seen below that each inner sum in (7.19) converges neatly, so that the separation of terms by the outer sum is justified. This formula will guide us towards the definition of the functions $q_{\sigma,\xi}(P, v \,|\, f)$.

In the following we assume that $s \in W/W_Q$ and $[s] = \sigma$. For $w \in \mathcal{W}$ we define the function $F_{s,w}\colon A^+_{\mathrm{q}}(P_1) \times \Omega \to V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}$ by

$$F_{s,w}(a, \lambda) = \sum_{\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1)} a^{-\mu} q_{s,\mu}(P_1, w \,|\, f, \log a, \lambda)(e),$$

for $a \in A^+_{\mathrm{q}}(P_1)$, $\lambda \in \Omega$.

The representation $\tilde{\tau} := 1 \otimes \tau$ of K on the complete locally convex space $\mathcal{O}(\Omega) \otimes V_\tau$ is smooth. We shall apply the results of Section 3, with $\tilde{\tau}$ in place of τ. The series defining $F_{s,w}$ is a $\Delta(P_1)$-exponential polynomial series with coefficients in $\mathcal{O}(\Omega) \otimes V_\tau$. By condition (c) of Definition 7.1 it converges neatly on $A^+_{\mathrm{q}}(P_1)$; hence $F_{s,w}$ may be viewed as an element of $C^{\mathrm{ep}}(A^+_{\mathrm{q}}(P_1), [\mathcal{O}(\Omega) \otimes V_\tau]^{K_{\mathrm{M}} \cap wHw^{-1}})$. In view of the isomorphism (2.9), there exists a unique function $F_s \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tilde{\tau})$ such that $T^{\downarrow}_{P_1,w}(F_s)(a) = F_s(aw) = F_{s,w}(a)$, for $w \in \mathcal{W}$ and $a \in A^+_{\mathrm{q}}(P_1)$. From the definition of F_s it follows that $\mathrm{Exp}(P_1, w \,|\, F_s) \subset sW_Q Y - \mathbb{N}\Delta(P_1)$, for every $w \in \mathcal{W}$. Moreover, for every $w \in \mathcal{W}$ and every $\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1)$,

$$q_{-\mu}(P_1, w \,|\, F_s, X, m)(\lambda) = q_{s,\mu}(P_1, w \,|\, f, X)(\lambda, m), \tag{7.20}$$

for $X \in \mathfrak{a}_{P\mathrm{q}}$, $m \in \mathrm{X}_{0,w}$ and $\lambda \in \Omega$. By transitivity of asymptotics, cf. Theorem 3.5, applied to F_s, we have that $\mathrm{Exp}(P, v \,|\, F_s)_{P_1,u} \subset \sigma \cdot Y - \mathbb{N}\Delta_r(P)$, for $u \in \mathcal{W}_{P,v}$. Moreover, by the same result it follows that, for $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$,

$$q_{-\xi}(P, v \,|\, F_s)(X, mbu) = \sum_{\substack{\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1) \\ \mu|_{\mathfrak{a}_{P\mathrm{q}}} = \xi}} b^{-\mu} q_{-\mu}(P_1, uv \,|\, F_s, X + \log b, m), \tag{7.21}$$

where the series on the right-hand side converges neatly as a $\Delta_P(P_1)$-exponential polynomial series in the variable $b \in {}^*A^+_{P\mathrm{q}}({}^*P_1)$, with coefficients in $C^\infty(\mathrm{X}_{0,uv} : \tilde{\tau}_{\mathrm{M}})$. In particular, the asserted convergence of (7.19) follows.

Substituting (7.20) in the right-hand side of (7.19) and using (7.21) we find, for $\lambda \in \Omega'$, that

$$q_{\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi}(P, v \,|\, f_\lambda, X, mbu) = \sum_{\substack{s \in W/W_Q \\ [s] = \sigma}} b^{s\lambda - \rho_{P_1}} q_{-\xi}(P, v \,|\, F_s)(X, mbu)(\lambda). \tag{7.22}$$

We are now ready to define the functions $q_{\sigma,\xi}(P, v \,|\, f)$.

Let 1 denote the trivial representation of K in $\mathbb{C}$, and 1_P its restriction to K_P. If $s \in W/W_Q$, we define the function $\varphi_s \in \mathcal{O}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, C^\infty(\mathrm{X}_{P,v,+} : 1_P))$ by

$$\varphi_s(\lambda, kbu) = b^{s\lambda - \rho_{P_1}}, \tag{7.23}$$

for $\lambda \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, $u \in \mathcal{W}_{P,v}$, $k \in K_P$ and $b \in {}^*A^+_{P\mathrm{q}}({}^*P_1)$. Moreover, for $\sigma \in W/\!\sim_{P|Q}$ and $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$ we define the function $q_{\sigma,\xi}(P, v \,|\, f) \colon \mathfrak{a}_{P\mathrm{q}} \times \Omega \to C^\infty(\mathrm{X}_{P,v,+} : \tau_P)$ by

$$q_{\sigma,\xi}(P, v \,|\, f, X, \lambda)(m) = \sum_{\substack{s \in W/W_Q \\ [s] = \sigma}} \varphi_s(\lambda, m)\, q_{-\xi}(P, v \,|\, F_s, X, m)(\lambda), \tag{7.24}$$

for $X \in \mathfrak{a}_{P\mathrm{q}}$, $\lambda \in \Omega$ and $m \in \mathrm{X}_{P,v,+}$.

If $1 < R \leq \infty$, then the locally convex space $C^\infty(\mathrm{X}_{P,v,+}[R], \mathcal{O}(\Omega) \otimes V_\tau)$ is naturally isomorphic with $\mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+}[R], V_\tau))$, see Appendix A. The isomorphism induces in turn a natural isomorphism of locally convex spaces

$$C^\infty(\mathrm{X}_{P,v,+}[R] : \tilde{\tau}_P)) \simeq \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+}[R] : \tau_P)). \tag{7.25}$$

In particular, for $R = \infty$, we obtain that $C^\infty(\mathrm{X}_{P,v,+} : \tilde{\tau}_P)$ is naturally isomorphic with $\mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+} : \tau_P))$. Thus, from (7.24) we deduce that $q_{\sigma,\xi}(P, v \,|\, f)$ is an element of $P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+} : \tau_P))$.

Combining (7.22), (7.23) and (7.24) we infer that (7.14) holds for $X \in \mathfrak{a}_{P\mathrm{q}}$, $\lambda \in \Omega'$. On the other hand, if $\lambda \in \Omega'$, then it follows from (3.9) with P and f_λ in place of Q and f, that, for $R > 1$, $m \in \mathrm{X}_{P,v,+}[R]$ and $a \in A^+_{P\mathrm{q}}(R^{-1})$,

$$f_\lambda(mav) = \sum_{\sigma \in W/\sim_{P|Q}} a^{\sigma\lambda - \rho_P} \sum_{\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)} a^{-\xi} q_{\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi}(P, v \,|\, f_\lambda, \log a)(m), \tag{7.26}$$

where the series converges neatly on $A^+_{P\mathrm{q}}(R^{-1})$, as a $\Delta_r(P)$-exponential polynomial series with coefficients in V_τ (use (7.16) and Lemma 6.2). Substituting (7.14) in (7.26) we obtain the identity (7.13) for $\lambda \in \Omega'$, $m \in \mathrm{X}_{P,v,+}[R]$ and $a \in A^+_{P\mathrm{q}}(R^{-1})$, with the convergence as asserted.

Thus, it remains to show that the identity (7.13) extends to all $\lambda \in \Omega$ and that the final assertion of the theorem holds. We will first establish the final assertion.

It follows from Theorem 3.5 that the series

$$\sum_{\xi \in -sW_Q Y|_{\mathfrak{a}_{P\mathrm{q}}} + \mathbb{N}\Delta_r(P)} a^{-\xi} q_{-\xi}(P, v \,|\, F_s, \log a) \tag{7.27}$$

converges neatly on $A^+_{P\mathrm{q}}(R^{-1})$ as a $\Delta_r(P)$-exponential polynomial series with coefficients in the space (7.25). The series (7.15) arises as the sum over $s \in W/W_Q$ with $[s] = \sigma$ of the series in (7.27) multiplied by φ_s. Since multiplication by φ_s induces a continuous linear endomorphism of the space (7.25), this establishes the final assertion of the theorem.

From the final assertion it follows that, for every $R > 1$, the series on the right-hand side of (7.13) defines a holomorphic function of $\lambda \in \Omega$, for every $m \in \mathrm{X}_{P,v,+}[R]$ and $a \in A^+_{P\mathrm{q}}(R^{-1})$. For such m, a the function $\lambda \mapsto f_\lambda(mav)$ is holomorphic in $\lambda \in \Omega$ by Lemma 7.4; hence the identity (7.13) extends to all $\lambda \in \Omega$, by density of Ω' in Ω. $\square$

Theorem 7.8 (Transitivity of asymptotics). *Let Q, Ω, Y, f, P and v be as in Theorem 7.7. Let $P_1 \in \mathcal{P}^{\min}_\sigma$ be contained in P. Let $\sigma \in W/\sim_{P|Q}$ and $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$. Then for every $X \in \mathfrak{a}_{P\mathrm{q}}$, all $u \in N_{K_P}(\mathfrak{a}_\mathrm{q})$, $b \in {}^*A^+_{P\mathrm{q}}({}^*P_1)$, $m \in M$ and $\lambda \in \Omega$,*

$$q_{\sigma,\xi}(P, v \,|\, f, X)(\lambda, mbu) = \sum_{\substack{s \in W/W_Q \\ [s]=\sigma}} b^{s\lambda - \rho_{P_1}} \sum_{\substack{\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1) \\ \mu|_{\mathfrak{a}_{P\mathrm{q}}} = \xi}} b^{-\mu}\, q_{s,\mu}(P_1, uv \,|\, f, X + \log b)(\lambda, m). \tag{7.28}$$

Moreover, for every $s \in W/W_Q$ with $[s] = \sigma$ and every $X \in \mathfrak{a}_{P\mathrm{q}}$, the series

$$\sum_{\substack{\mu \in -sW_Q Y + \mathbb{N}\Delta(P_1) \\ \mu|_{\mathfrak{a}_{P\mathrm{q}}} = \xi}} b^{-\mu}\, q_{s,\mu}(P_1, uv \,|\, f, X + \log b) \tag{7.29}$$

converges neatly on ${}^*A^+_{P\mathrm{q}}({}^*P_1)$ as a $\Delta(P_1)$-exponential polynomial series in the variable b with coefficients in $\mathcal{O}(\Omega, C^\infty(\mathrm{X}_{0,uv} : \tau_\mathrm{M}))$.

Proof: Fix $u \in N_{K_P}(\mathfrak{a}_\mathrm{q})$. Moreover, we fix a set $\mathcal{W}_{P,v}$ as in the beginning of the proof of Theorem 7.7 such that it contains the element u. We will also use the remaining notation of the proof of the mentioned theorem.

Using (7.20) we see that, via the natural isomorphism of $\mathcal{O}(\Omega, C^\infty(\mathrm{X}_{0,uv} : \tau_{\mathrm{M}}))$ with $C^\infty(\mathrm{X}_{0,uv} : \tilde{\tau}_{\mathrm{M}})$, the series (7.29) may be identified with the series with coefficients in $C^\infty(\mathrm{X}_{0,uv} : \tilde{\tau}_{\mathrm{M}})$ that arises from the series on the right-hand side of (7.21) by omitting the evaluation at m. The neat convergence of the latter series was noted already. Moreover, the identity (7.28) follows by insertion of (7.21) in the definition (7.24) of $q_{\sigma,\xi}$. □

The following result is an important consequence of 'holomorphy of asymptotics.'

Lemma 7.9 *Let $Q \in \mathcal{P}_\sigma$, $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset and $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset. Let $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ and let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$, and $\sigma \in W/{\sim_{P|Q}}$.*

Let $\xi \in -\sigma{\cdot}Y + \mathbb{N}\Delta_r(P)$ and assume that there exists a $\lambda_0 \in \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY) \cap \Omega$ such that

$$\sigma\lambda_0|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi \in \mathrm{Exp}(P, v \,|\, f_{\lambda_0}). \tag{7.30}$$

Then there exists a full open subset Ω_0 of Ω such that

$$\sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \xi \in \mathrm{Exp}(P, v \,|\, f_\lambda), \qquad (\forall \lambda \in \Omega_0).$$

Proof: From (7.30) combined with (7.14) it follows that the $P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes C^\infty(\mathrm{X}_{P,v,+} : \tau_P)$-valued holomorphic function $q \colon \lambda \mapsto q_{\sigma,\xi}(P, v \,|\, f, \,\cdot\,, \lambda)$ does not vanish at $\lambda = \lambda_0$. Hence there exists a full open subset $\Omega_1 \subset \Omega$ such that $q(\lambda) \neq 0$ for all $\lambda \in \Omega$. Let $\Omega_0 \colon= \Omega_1 \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY)$, then the conclusion follows by application of (7.14).
□

We end this section with a result describing the behavior of the functions $q_{\sigma,\xi}$ under the action of $N_K(\mathfrak{a}_\mathrm{q})$. Let $Q, P \in \mathcal{P}_\sigma$ and $u \in N_K(\mathfrak{a}_\mathrm{q})$, and put $P' = uPu^{-1}$. The left multiplication by u naturally induces a map $W/{\sim_{P|Q}} \to W/{\sim_{P'|Q}}$, which we denote by $\sigma \mapsto u\sigma$. Moreover, the endomorphism $\mathrm{Ad}(u^{-1})^*$ of $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ restricts to a linear map $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}} \to \mathfrak{a}^*_{P'\mathrm{q}\mathbb{C}}$, which we denote by $\eta \mapsto u\eta$. With these notations, if $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a finite subset and $\sigma \in W/{\sim_{P|Q}}$, then

$$u(\sigma \cdot Y) = (u\sigma) \cdot Y;$$

see also (7.12). For $v \in N_K(\mathfrak{a}_\mathrm{q})$, let the map $\rho_{\tau,u} \colon C^\infty(\mathrm{X}_{P,v,+} : \tau_P) \to C^\infty(\mathrm{X}_{P',uv,+} : \tau_{P'})$ be defined by (3.24) with P in place of Q.

If $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is an open subset, let $\mathrm{Ad}(u^{-1})^* \otimes 1 \otimes \rho_{\tau,u}$ denote the naturally induced map from $P(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+} : \tau_P))$ to $P(\mathfrak{a}_{P'\mathrm{q}}) \otimes \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P',uv,+} : \tau_{P'}))$.

Lemma 7.10 *Let $Q \in \mathcal{P}_\sigma$, $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset and $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset. Let $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. If $P \in \mathcal{P}_\sigma$ and $u, v \in N_K(\mathfrak{a}_\mathrm{q})$, then for all $\sigma \in W/{\sim_{P|Q}}$ and $\xi \in \sigma \cdot Y$,*

$$q_{u\sigma,u\xi}(uPu^{-1}, uv \,|\, f) = [\mathrm{Ad}(u^{-1})^* \otimes 1 \otimes \rho_{\tau,u}]\, q_{\sigma,\xi}(P, v \,|\, f).$$

Proof: From combining (7.14) and Lemma 3.6 it follows that there exists a full open subset Ω_0 of Ω such that, for $\lambda \in \Omega_0$,

$$q_{u\sigma,u\xi}(uPu^{-1}, uv \,|\, f, \,\cdot\,, \lambda) = [\mathrm{Ad}(u^{-1})^* \otimes \rho_{\tau,u}] q_{\sigma,\xi}(P, v \,|\, f, \,\cdot\,, \lambda).$$

The result now follows by holomorphy of the above expressions in λ and density of Ω_0. □

8 Asymptotic globality

In this section we introduce the notion of asymptotic globality of a spherical function on X_+ and of an analytic family of such functions. We discuss properties needed in the statement and proof of the vanishing theorem in the next section.

Definition 8.1 *Let $P \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$. A function $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ is said to be asymptotically global along (P, v) at an element $\xi \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ if, for every $X \in \mathfrak{a}_{P\mathrm{q}}$, the V_τ-valued smooth function $q_\xi(P, v \,|\, f, X)$ has a C^∞-extension from $\mathrm{X}_{P,v,+}$ to $\mathrm{X}_{P,v}$.*

Remark 8.2 Since $q_\xi(P, v \,|\, f, X)$ is polynomial in X, with values in $C^\infty(\mathrm{X}_{P,v,+} : \tau_P)$, the requirement on q_ξ implies that $q_\xi(P, v \,|\, f)$ is a polynomial $C^\infty(\mathrm{X}_{P,v} : \tau_P)$-valued function on $\mathfrak{a}_{P\mathrm{q}}$.

Note that for P minimal the condition of asymptotic globality along (P, v) is automatically fulfilled, since $\mathrm{X}_{P,v,+} = \mathrm{X}_{P,v}$.

Finally, if $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$, then f is asymptotically global along (G, e) at every exponent if and only if f extends smoothly to X (use Remark 1.6).

The property of asymptotic globality is preserved under the action of $\mathbb{D}(\mathrm{X})$ in the following fashion. If $P \in \mathcal{P}_\sigma$, then by $\preceq_{\Delta_r(P)}$ we denote the partial ordering on $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, defined as in (1.6), with $\mathfrak{a}_{P\mathrm{q}}$ and $\Delta_r(P)$ in place of $\mathfrak{a}$ and Δ, respectively.

Proposition 8.3 *Let $f \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ and $D \in \mathbb{D}(\mathrm{X})$. Let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$ and $\xi_0 \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$. If f is asymptotically global along (P, v) at every exponent $\xi \in \mathrm{Exp}(P, v \,|\, f)$ with $\xi_0 \preceq_{\Delta_r(P)} \xi$, then Df is asymptotically global along (P, v) at ξ_0.*

Proof: Let $u := \mu'_P(D) + u_+$ be the element of $\mathcal{D}_{1P}$ associated with D as in Proposition 4.10, with P in place of Q. The key idea in the present proof is that u has a $\Delta_r(P)$-exponential polynomial expansion with coefficients that are globally defined smooth functions on $M_{P\sigma}$, by Cor. 4.9. More precisely, the expansion $\mathrm{ep}(u)$ is a finite sum, as i ranges over a finite index set I, of terms of the form

$$\mathrm{ep}(u)_i = \sum_{\nu \in \mathbb{N}\Delta_r(P)} a^{-\nu} \varphi_{i,\nu} \otimes S_{i,\nu} \otimes u_{i,\nu} \otimes v_{i,\nu}.$$

Here $\varphi_{i,\nu} \in C^\infty(M_{P\sigma})$, $S_{i,\nu} \in \mathrm{End}(V_\tau)$, $u_{i,\nu} \in U(\mathfrak{m}_{P\sigma})$ and $v_{i,\nu} \in U(\mathfrak{a}_{P\mathrm{q}})$, and $\deg(u_{i,\nu}) + \deg(v_{i,\nu}) \le \deg(D)$ for all i, ν. By Lemma 4.12, Df belongs to $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$ and its (P, e)-expansion results from the (P, e)-expansion of f by the formal application of the element $\mathrm{ep}(u)$. Hence the asymptotic coefficient of ξ_0 is given by the finite sum

$$q_{\xi_0}(P, e \,|\, Df)(X, m) = \sum_{\substack{\xi \in \mathrm{Exp}(P,e\,|\,f) \\ \nu \in \mathbb{N}\Delta_r(P) \\ \xi - \nu = \xi_0}} \sum_{i \in I} \varphi_{i,\nu}(m) S_{i,\nu}[\, q_\xi(P, e \,|\, f)(X; T_\xi(v_{i,\nu}), m; u_{i,\nu})\,].$$

Let now f satisfy the hypothesis of the proposition. The $\xi' s$ occurring in the above sum belong to $\xi_0 + \mathbb{N}\Delta_r(P)$, hence satisfy $\xi_0 \preceq_{\Delta_r(P)} \xi$. By hypothesis, the associated coefficients $q_\xi(P, e \,|\, f)$ all extend smoothly to $\mathfrak{a}_{P\mathrm{q}} \times M_{P\sigma}$, see Remark 8.2. Therefore, so does $q_{\xi_0}(P, e \,|\, Df)$. This establishes the result for arbitrary $P \in \mathcal{P}_\sigma$ and the special choice $v = e$. The result with general $v \in N_K(\mathfrak{a}_\mathrm{q})$ now follows by application of Lemma 3.6 (cf. Lemma 8.7 (a)). □

We shall also introduce a notion of asymptotic globality for families from the space $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ introduced in the previous section, with $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ an open subset.

Definition 8.4 *Let $Q \in \mathcal{P}_\sigma$, Y a finite subset of ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset. Let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$ and $\sigma \in W/\!\sim_{P|Q}$.*

We will say that a family $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ is σ-global along (P, v), if there exists a dense open subset Ω_0 of Ω, such that, for every $\lambda \in \Omega_0$, the function f_λ is asymptotically global along (P, v) at each exponent $\xi \in \sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + \sigma \cdot Y - \rho_P - \mathbb{N}\Delta_r(P)$.

Remark 8.5 If Y_1 and Y_2 are finite subsets of ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ with $Y_1 \subset Y_2$, then obviously

$$C^{\mathrm{ep}}_{Q,Y_1}(\mathrm{X}_+ : \tau : \Omega) \subset C^{\mathrm{ep}}_{Q,Y_2}(\mathrm{X}_+ : \tau : \Omega).$$

If f belongs to the first of these spaces, then the condition of σ-globality along (P, v) relative to Y_1 is equivalent to the similar condition relative to Y_2. This is readily seen by using Lemmas 6.2 and 6.3. From this we see that the notion of σ-globality along (P, v) extends to the space

$$C^{\mathrm{ep}}_{Q}(\mathrm{X}_+ : \tau : \Omega) := \bigcup_{Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \text{ finite}} C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$$

The property of asymptotic globality for families is also stable under the action of $\mathbb{D}(\mathrm{X})$.

Corollary 8.6 *Let $Q \in \mathcal{P}_\sigma$, Y a finite subset of ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset. Let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$ and $\sigma \in W/\!\sim_{P|Q}$.*

Let $f \in C^{\rm ep}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ be σ-global along (P, v). Then for every $D \in \mathbb{D}(\mathrm{X})$ the family $Df \in C^{\rm ep}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ is σ-global along (P, v) as well.

Proof: It follows from Proposition 7.6 that Df belongs to $C^{\rm ep}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. According to Theorem 7.7, both sets $\mathrm{Exp}(P, v \,|\, f_\lambda)$ and $\mathrm{Exp}(P, v \,|\, Df_\lambda)$ are contained in the set $E_\lambda\colon = W(\lambda + Y)|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \mathbb{N}\Delta_r(P)$, for every $\lambda \in \Omega$.

Let Ω_0 be as in Definition 8.4. Then the set $\Omega_0'\colon = \Omega_0 \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY)$ is open dense in Ω by Lemma 6.3. Let $\lambda \in \Omega_0'$ and let $\xi_0 \in \sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + \sigma \cdot Y - \rho_P - \mathbb{N}\Delta_r(P)$. If $\xi \in \mathrm{Exp}(P, v \,|\, f_\lambda)$ satisfies $\xi_0 \preceq \xi$, then $\xi \in \sigma\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + \sigma \cdot Y - \rho_P - \mathbb{N}\Delta_r(P)$ by Lemma 6.2. By hypothesis, f_λ is asymptotically global along (P, v) at the exponent ξ. It now follows by application of Proposition 8.3 that Df_λ is asymptotically global along (P, v) at ξ_0. $\square$

The following lemma describes the behavior of asymptotic globality under the action of $N_K(\mathfrak{a}_\mathrm{q})$.

Lemma 8.7 *Let $P \in \mathcal{P}_\sigma$ and $u, v \in N_K(\mathfrak{a}_\mathrm{q})$. Put $P' = uPu^{-1}$ and $v' = uv$.*

(a) *Let $f \in C^{\rm ep}(\mathrm{X}_+ : \tau)$ and $\xi \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$. If f is asymptotically global along (P, v) at ξ, then f is asymptotically global along (P', v') at $u\xi$.*

(b) *Let $Q \in \mathcal{P}_\sigma$, $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset, $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset, $f \in C^{\rm ep}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ and $\sigma \in W/\sim_{P|Q}$. If f is σ-global along (P, v), then f is $u\sigma$-global along (P', v').*

Proof: From (3.24) with P in place of Q it is readily seen that $\rho_{\tau,u}$ maps $C^\infty(\mathrm{X}_{P,v} : \tau_P)$ to $C^\infty(\mathrm{X}_{P',v'} : \tau_{P'})$. Then (a) and (b) follow immediately from Lemmas 3.6 and 7.10, respectively. $\square$

We end this section with the following result, which shows that the globality condition is fulfilled for a certain natural class of τ-spherical functions. From the text preceding Lemma 5.5 we recall that $\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}_\mathrm{q}$ and that if $\mu \in \mathfrak{b}^*_\mathbb{C}$, then by I_μ we denote the kernel of the character $\gamma(\,\cdot : \mu)$ of $\mathbb{D}(\mathrm{X})$. Thus I_μ is an ideal in $\mathbb{D}(\mathrm{X})$ of codimension one (over $\mathbb{C}$).

Proposition 8.8 *Let $\mu \in \mathfrak{b}^*_\mathbb{C}$ and let $f \in \mathcal{E}(\mathrm{X} : \tau : I_\mu)$. Then $f|_{\mathrm{X}_+} \in C^{\rm ep}(\mathrm{X}_+ : \tau)$. Moreover, this function is asymptotically global along all pairs $(P, v) \in \mathcal{P}_\sigma \times N_K(\mathfrak{a}_\mathrm{q})$ and at all exponents $\xi \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.*

Proof: The first statement follows immediately from Lemma 5.3. By Lemma 8.7 (a) it suffices to consider $v = e$ and arbitrary $P \in \mathcal{P}_\sigma$. Let $\psi \in V_\tau$ be fixed. Then it suffices to prove that the scalar valued function $m \mapsto q_\xi(X, m)\colon = \langle q_\xi(P, e|f, X, m) \,|\, \psi\rangle$ on $\mathrm{X}_{P,+}$ has a C^∞ extension to X_P, for each $\xi \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, $X \in \mathfrak{a}_{P\mathrm{q}}$. It follows from Theorem 3.4 that

$$\langle f(ma) \,|\, \psi\rangle = \sum_{\xi \in Y - \mathbb{N}\Delta_r(P)} a^\xi q_\xi(\log a, m). \tag{8.1}$$

On the other hand, it follows from [5], Lemma 12.3, that [5], Thm. 12.8 can be applied to the K-finite function $F\colon x \mapsto \langle f(x) \,|\, \psi \rangle$. By uniqueness of asymptotics (see Lemma 1.7 and its proof) the expansion (8.1) coincides with that of [5], Thm. 12.8. We conclude that, in the notation of loc. cit., $q_\xi(X, m) = p_{\mu|_{\mathfrak{a}_\mathrm{q}},\xi}(P|F, m, X)$ for all $X \in \mathfrak{a}_{P\mathrm{q}}$, $m \in \mathrm{X}_{P,+}$. The function $x \mapsto p_{\mu|_{\mathfrak{a}_\mathrm{q}},\xi}(P|F, x, X)$ is smooth on G. From this the smooth extension of $q_\xi(X, m)$ follows immediately. $\square$

9 A vanishing theorem

In this section we formulate and prove the vanishing theorem. We assume that Q is a σ-parabolic subgroup containing A_q.

As before, let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{q}$ containing $\mathfrak{a}_\mathrm{q}$. By ${}^*\mathfrak{a}_{Q\mathrm{q}}$ and ${}^*\mathfrak{b}_Q$ we denote the orthocomplements of $\mathfrak{a}_{Q\mathrm{q}}$ in $\mathfrak{a}_\mathrm{q}$ and $\mathfrak{b}$, respectively. Let $\mathfrak{b}_\mathrm{k} \colon= \mathfrak{b} \cap \mathfrak{k}$; then

$$ {}^*\mathfrak{b}_Q = \mathfrak{b}_\mathrm{k} \oplus {}^*\mathfrak{a}_{Q\mathrm{q}}. $$

We write D_Q for the collection of functions $\delta\colon {}^*\mathfrak{b}_{Q\mathbb{C}}^* \to \mathbb{N}$ with finite support $\operatorname{supp}\delta$. For $\delta \in \mathrm{D}_Q$ we put

$$ |\delta| \colon= \sum_{\nu \in \operatorname{supp}\delta} \delta(\nu). $$

For $\delta \in \mathrm{D}_Q$ and $\lambda \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ we define the ideal $I_{\delta,\lambda}$ in $\mathbb{D}(\mathrm{X})$ as the following product of ideals

$$ I_{\delta,\lambda} \colon= \prod_{\nu \in \operatorname{supp}\delta} (I_{\nu+\lambda})^{\delta(\nu)}. \tag{9.1} $$

If $\delta = 0$, this ideal is understood to be the full ring $\mathbb{D}(\mathrm{X})$. Being a product of cofinite ideals in the Noetherian ring $\mathbb{D}(\mathrm{X})$, the ideal $I_{\delta,\lambda}$ is cofinite.

Definition 9.1 *Let $\Omega \subset \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ be a non-empty open subset and $\delta \in \mathrm{D}_Q$. For every finite subset $Y \subset {}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ we define*

$$ \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta) \tag{9.2} $$

to be the space of families $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ (cf. Def. 7.1) such that for every $\lambda \in \Omega$ the function $f_\lambda \colon x \mapsto f(\lambda, x)$ is annihilated by the cofinite ideal (9.1). Moreover, we define

$$ \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta) \colon= \bigcup_{Y \subset {}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \text{ finite}} \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta). $$

Note that the space (9.2) depends on Q through its σ-split component $A_{Q\mathrm{q}}$. If $\nu \in {}^*\mathfrak{b}_{Q\mathbb{C}}^*$, we denote by δ_ν the characteristic function of the set $\{\nu\}$. Then

$\delta_\nu \in \mathrm{D}_Q$. Moreover, if $\delta \in \mathrm{D}_Q$ and $\nu \in \operatorname{supp}\delta$, then $\delta - \delta_\nu \in \mathrm{D}_Q$ and $|\delta - \delta_\nu| = |\delta| - 1$.

Lemma 9.2 *Let* $f \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)$.

(a) *If* $D \in \mathbb{D}(\mathrm{X})$ *then* $Df \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)$.

(b) *If* $D \in \mathbb{D}(\mathrm{X})$ *and* $\nu \in \operatorname{supp}\delta$ *then the function* $g\colon \Omega \times \mathrm{X}_+ \to V_\tau$ *defined by*

$$g(\lambda, x) := [D - \gamma(D : \nu + \lambda)]f_\lambda(x), \qquad (\lambda \in \Omega,\ x \in \mathrm{X}_+), \tag{9.3}$$

belongs to $\mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta - \delta_\nu)$.

Proof: Let $D \in \mathbb{D}(\mathrm{X})$. By Proposition 7.6, the family Df belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. Moreover, if $\lambda \in \Omega$ and $D' \in I_{\delta,\lambda}$, then $D'(Df)_\lambda = D'Df_\lambda = DD'f_\lambda = 0$ and we see that assertion (a) holds.

The function $\lambda \mapsto \gamma(D : \nu + \lambda)$ is polynomial on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, hence holomorphic on Ω and it follows that $G\colon (\lambda, x) \mapsto \gamma(D : \nu + \lambda) f(\lambda, x)$ belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. Hence $g = Df - G$ belongs to the latter space as well. Furthermore, if $D' \in I_{\delta - \delta_\nu, \lambda}$, then $D'' := D'(D - \gamma(D : \nu + \lambda)) \in I_{\delta,\lambda}$, and we see that $D'g_\lambda = D''f_\lambda = 0$. Hence (b) holds. $\square$

Remark 9.3 It follows from Lemma 9.2 (a) that (7.8) defines a representation of $\mathbb{D}(\mathrm{X})$ in $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)$, leaving the subspaces $\mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)$ invariant.

Lemma 9.4 *Let* $Q \in \mathcal{P}_\sigma$, $\delta \in \mathrm{D}_Q$ *and* Ω *a connected non-empty open subset of* $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. *Assume that* $f \in C^{\mathrm{ep}}_Q(\mathrm{X}_+ : \tau : \Omega)$. *If* f_λ *is annihilated by* $I_{\delta,\lambda}$ *for* λ *in a non-empty open subset* Ω' *of* Ω, *then* $f \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)$.

Proof: Fix a finite subset $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. We proceed by induction on $|\delta|$.

First, assume that $|\delta| = 0$. Then $I_{\delta,\lambda} = \mathbb{D}(\mathrm{X})$ for all λ and hence $f|_{\Omega' \times \mathrm{X}_+} = 0$. Since Ω is connected, this implies that $f = 0$, see Lemma 7.4.

Next assume that $|\delta| = k \geq 1$ and assume the result has already been established for all $\delta \in \mathrm{D}_Q$ with $|\delta| < k$. Fix $\nu \in \operatorname{supp}\delta$ and put $\delta' = \delta - \delta_\nu$, then $|\delta'| < k$. Let $D \in \mathbb{D}(\mathrm{X})$ and define g as in (9.3). Then $g \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$, as seen in the proof of Lemma 9.2. On the other hand, it follows from (b) of that lemma that $g|_{\Omega' \times \mathrm{X}_+} \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega' : \delta')$. Hence $g \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta')$ by the induction hypothesis. Fix $\lambda \in \Omega$. Then it follows, for $D' \in I_{\delta',\lambda}$, that $D'(D - \gamma(D : \nu + \lambda))f_\lambda = D'g_\lambda = 0$. Since D was arbitrary, we conclude that f_λ is annihilated by the ideal $I_{\delta',\lambda} I_{\nu+\lambda} = I_{\delta,\lambda}$. $\square$

We define the following subset of $\mathcal{P}_\sigma$, consisting of the parabolic subgroups whose σ-split rank is of codimension one,

$$\mathcal{P}_\sigma^1 := \{P \in \mathcal{P}_\sigma \mid \dim \mathfrak{a}_{\mathrm{q}} / \dim \mathfrak{a}_{P\mathrm{q}} = 1\}.$$

Definition 9.5 *Let $Q \in \mathcal{P}_\sigma$, $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset and $\delta \in \mathrm{D}_Q$. By $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ we denote the space of functions $f \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)$ satisfying the following condition.*

> *For every $s \in W$ and every $P \in \mathcal{P}_\sigma^1$ with $s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{P\mathrm{q}}$, the family f is $[s]$-global along (P, v), for all $v \in N_K(\mathfrak{a}_{\mathrm{q}})$; here $[s]$ denotes the image of s in $W/{\sim_{P|Q}} = W_P \backslash W / W_Q$.*

If $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a finite subset, we define

$$\mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}} := \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta) \cap \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}.$$

Remark 9.6 Note that $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ depends on Q through its σ-split component $\mathfrak{a}_{Q\mathrm{q}}$.

The equality $W/{\sim_{P|Q}} = W_P \backslash W / W_Q$ follows from Lemma 6.5. Note that the condition $s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{P\mathrm{q}}$ on s factors to a condition on its class in $W_P \backslash W / W_Q$.

The following result reduces the globality condition of Definition 9.5 to a condition involving a smaller set of $(s, P) \in W \times \mathcal{P}_\sigma^1$. Its formulation requires some more notation.

Let Δ be a fixed basis for the root system Σ, let Σ^+ be the associated system of positive roots and $\mathfrak{a}_{\mathrm{q}}^+$ the associated open positive chamber. Let P_0 be the unique element of $\mathcal{P}_\sigma^{\min}$ with $\Delta(P_0) = \Delta$. A σ-parabolic subgroup Q is said to be standard if it contains P_0; of course then $Q \in \mathcal{P}_\sigma$. Given such a Q, we write Δ_Q for the subset of Δ consisting of the roots vanishing on $\mathfrak{a}_{Q\mathrm{q}}$ and $\Delta(Q)$ for its complement.

If α is any root in Δ, we write $\mathfrak{n}_\alpha$ for the sum of the root spaces $\mathfrak{g}_\beta$ where β ranges over the set $\Sigma^+ \setminus \mathbb{N}\alpha$. Moreover, we put $N_\alpha := \exp(\mathfrak{n}_\alpha)$ and write $M_{1\alpha}$ for the centralizer in G of the root hyperplane $\ker \alpha$. Then $P_\alpha = M_{1\alpha} N_\alpha$ is the standard parabolic subgroup with $\Delta_{P_\alpha} = \{\alpha\}$. We write $P_\alpha = M_\alpha A_\alpha N_\alpha$ and $P_\alpha = M_{\sigma\alpha} A_{\alpha\mathrm{q}} N_\alpha$ for the Langlands and σ-Langlands decompositions of P_α, respectively. Accordingly, $\mathfrak{a}_{\alpha\mathrm{q}} = \ker \alpha$ and ${}^*\mathfrak{a}_{\alpha\mathrm{q}} = (\ker \alpha)^\perp$. Finally, we write $W_\alpha = W_{P_\alpha}$ for the centralizer of $\ker \alpha$ in W.

Lemma 9.7 *Let $Q \in \mathcal{P}_\sigma$ be a standard parabolic subgroup, $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset, $\delta \in \mathrm{D}_Q$ and $f \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)$. Then f belongs to $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ if and only if the following condition is fulfilled.*

> *For every $s \in W$ and every $\alpha \in \Delta$ with $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$, the family f is $[s]$-global along (P_α, v), for all $v \in N_K(\mathfrak{a}_{\mathrm{q}})$; here $[s]$ denotes the image of s in $W/{\sim_{P_\alpha|Q}} = W_\alpha \backslash W / W_Q$.*

Proof: We must show that the condition of Definition 9.5 is fulfilled if and only if the above condition holds. For this we first observe that for $\alpha \in \Delta$ and $s \in W$,

$$s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0 \iff s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{\alpha\mathrm{q}}.$$

The 'only if part' is now immediate. For the 'if part', assume that the above condition is fulfilled. Let $(s, P) \in W \times \mathcal{P}_\sigma^1$ be such that $s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{P\mathrm{q}}$. There exist $\alpha \in \Delta$ and $t \in W$ such that $tPt^{-1} = P_\alpha$. It follows that $ts(\mathfrak{a}_{Q\mathrm{q}}) \not\subset t\mathfrak{a}_{P\mathrm{q}} = \ker \alpha$, hence $(ts)^{-1}\alpha = \alpha \circ (ts)$ is not identically zero on $\mathfrak{a}_{Q\mathrm{q}}$. From the hypothesis it now follows that f is $[ts]$-global along (tPt^{-1}, v), for all $v \in N_K(\mathfrak{a}_\mathrm{q})$. By Lemma 8.7 it follows that f is $[s]$-global along (P, w), for all $w \in N_K(\mathfrak{a}_\mathrm{q})$. $\square$

Lemma 9.8 *Let $Q \in \mathcal{P}_\sigma$, $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a non-empty open subset and $\delta \in \mathrm{D}_Q$. Then the space $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ is $\mathbb{D}(\mathrm{X})$-invariant. Moreover, $\mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ is a $\mathbb{D}(\mathrm{X})$-submodule, for every finite subset $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.*

Proof: This follows from combining the $\mathbb{D}(\mathrm{X})$-invariance of the space $\mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)$ with Proposition 8.3. $\square$

Definition 9.9 *Let $Q \in \mathcal{P}_\sigma$. An open subset Ω of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ will be called Q-distinguished if it is connected and if for every $\alpha \in \Sigma(Q)$ the function $\lambda \mapsto \langle \mathrm{Re}\,\lambda\,,\,\alpha\rangle$ is not bounded from above on Ω.*

In particular, a connected open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is Q-distinguished. In the following theorem we assume that ${}^Q\mathcal{W} \subset N_K(\mathfrak{a}_\mathrm{q})$ is a complete set of representatives for $W_Q\backslash W/W_{K\cap H}$.

Theorem 9.10 (Vanishing theorem). *Let $Q \in \mathcal{P}_\sigma$ and $\delta \in \mathrm{D}_Q$. Let $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a Q-distinguished open subset and let $f \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$. Assume that there exists a non-empty open subset $\Omega' \subset \Omega$ such that, for each $v \in {}^Q\mathcal{W}$,*

$$\lambda - \rho_Q \notin \mathrm{Exp}(Q, v\,|\,f_\lambda), \qquad (\lambda \in \Omega'). \tag{9.4}$$

Then $f = 0$.

The proof of this theorem will be given after the following lemmas on which it is based. We may and shall assume that Q is standard. Thus, Q contains the minimal standard σ-parabolic subgroup P_0 which will be denoted by P in the rest of this section.

Lemma 9.11 *Let $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a non-empty connected open subset, $\delta \in \mathrm{D}_Q$ and assume that $|\delta| = 1$. Let $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a finite subset and let $f \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)$. Moreover, let $v \in N_K(\mathfrak{a}_\mathrm{q})$ and assume that there exist $t \in W_Q$, $\eta \in Y$, $\mu \in \mathbb{N}\Delta$ and $u \in N_{K_Q}(\mathfrak{a}_\mathrm{q})$ such that*

$$\lambda + t\eta - \rho - \mu \in \mathrm{Exp}(P, uv\,|\,f_\lambda) \tag{9.5}$$

for λ in some non-empty open subset of Ω. Then there exists a full open subset $\Omega_0 \subset \Omega$ such that

$$\lambda - \rho_Q \in \mathrm{Exp}(Q, v \,|\, f_\lambda), \qquad (\lambda \in \Omega_0).$$

Proof: Let $\nu \in {}^*\mathfrak{b}_{Q\mathbb{C}}^*$ be the unique element such that $\mathrm{supp}\,\delta = \{\nu\}$. Fix t, η, μ and u with the mentioned property. Replacing μ by a $\preceq_\Delta$-smaller element if necessary we may in addition assume that μ is $\preceq_\Delta$-minimal subject to the condition that (9.5) holds for λ in some non-empty open subset of Ω. By holomorphy of asymptotics, see Lemma 7.9, it follows that (9.5) holds for λ in a full open subset Ω' of Ω. Moreover, using the minimality of μ and applying Lemma 6.2 we see that for every λ in the full open subset $\Omega_0 := \Omega' \cap \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P, WY)$ of Ω,

$$\lambda + t\eta - \rho - \mu \in \mathrm{Exp_L}(P, uv \,|\, f_\lambda).$$

Since f_λ is annihilated by $I_{\delta,\lambda} = I_{\nu+\lambda}$, this implies, in view of Lemma 5.5, that there exists a finite subset $\mathcal{L} \subset \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$ such that

$$\nu + \lambda \in W(\mathfrak{b})(\mathcal{L} + \lambda + t\eta - \mu), \qquad (\lambda \in \Omega_0).$$

For $\Lambda_0 \in \mathcal{L}$, $w \in W(\mathfrak{b})$ we define $\Omega_0(\Lambda_0, w)$ to be the set of $\lambda \in \Omega_0$ satisfying

$$\nu + \lambda = w(\Lambda_0 + \lambda + t\eta - \mu). \tag{9.6}$$

The union of these sets, as $\Lambda_0 \in \mathcal{L}$, $w \in W(\mathfrak{b})$, equals Ω_0. By finiteness of the union, we may select Λ_0 and w such that $\Omega_0(\Lambda_0, w)$ has a non-empty interior in Ω_0. Since $\Omega_0(\Lambda_0, w)$ is also the intersection of Ω_0 with an affine linear subspace of $\mathfrak{b}_{\mathbb{C}}^*$, it must be all of Ω_0. Hence for all $\lambda_1, \lambda_2 \in \Omega_0$ we have $w(\lambda_1 - \lambda_2) = \lambda_1 - \lambda_2$. Since Ω_0 is a non-empty open subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ this implies that w belongs to $W_Q(\mathfrak{b})$, the centralizer of $\mathfrak{a}_{Q\mathrm{q}}$ in $W(\mathfrak{b})$. From (9.6) we now deduce that $-w\mu = \nu - w\Lambda_0 - wt\eta$. The expression on the right-hand side of this equality has zero restriction to $\mathfrak{a}_{Q\mathrm{q}}$. Therefore, so has $w\mu$, and we conclude that also $\mu|_{\mathfrak{a}_{Q\mathrm{q}}} = 0$. Combining this fact with (9.5) and transitivity of asymptotics, see Theorem 3.5, we conclude that

$$\lambda - \rho_Q = [\lambda + t\eta - \rho - \mu]|_{\mathfrak{a}_{Q\mathrm{q}}} \in \mathrm{Exp}(Q, v \,|\, f_\lambda),$$

for all $\lambda \in \Omega_0$. □

For the formulation of the next lemma, we need the following definition.

Definition 9.12 *Let $\Omega \subset \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and $s_0 \in W$ be given. The subset $W(\Omega, s_0)$ of W is defined as follows. Let $s' \in W$. Then $s' \in W(\Omega, s_0)$ if and only if there exists a chain $s_1, \ldots, s_k = s'$ of elements in W, with $s_j s_{j-1}^{-1} = s_{\alpha_j}$ a simple reflection, such that the following condition (9.7) holds for each of the pairs $(s, \alpha) = (s_{j-1}, \alpha_j) \in W \times \Delta$, $j = 1, \ldots, k$.*

If $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$ then $\lambda \mapsto \mathrm{Re}\,\langle s\lambda\,,\,\alpha\rangle$ is not bounded from below on Ω. (9.7)

Notice that if Ω is dense in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then $W(\Omega, s_0) = W$ for all $s_0 \in W$. Indeed, (9.7) is then fulfilled by all elements $\alpha \in \Delta$. Hence, in order to verify the conditions of Definition 9.12 for $s' \in W$ arbitrary, we may choose as $s_{\alpha_1}, \dots, s_{\alpha_k}$ the elements in a reduced expression $s's_0^{-1} = s_{\alpha_k} \cdots s_{\alpha_1}$.

Lemma 9.13 *Let $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a non-empty connected open subset, $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset, and $\delta \in \mathrm{D}_Q$. Let $f \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ and $s \in W$. Assume that there exist $t \in W_Q$, $\eta \in Y$, $\mu \in \mathbb{N}\Delta$ and $w \in N_K(\mathfrak{a}_\mathrm{q})$ such that*

$$s\lambda + st\eta - \rho - \mu \in \mathrm{Exp}(P, w \,|\, f_\lambda), \tag{9.8}$$

for all λ in some non-empty open subset of Ω. Then for every $s_1 \in W(\Omega, s)$ there exist $t_1 \in W_Q$, $\eta_1 \in Y$, $\mu_1 \in \mathbb{N}\Delta$ and $w_1 \in N_K(\mathfrak{a}_\mathrm{q})$, such that

$$s_1\lambda + s_1t_1\eta_1 - \rho - \mu_1 \in \mathrm{Exp}(P, w_1 \,|\, f_\lambda), \tag{9.9}$$

for all λ in a full open subset of Ω. In particular, if Ω is dense in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then the above conclusion holds for every $s_1 \in W$.

Proof: In the proof we will frequently use the following consequence of Lemma 7.9, based on holomorphy of asymptotics. If $s_1 \in W$, $t_1 \in W_Q$, $\eta_1 \in Y$, $\mu_1 \in \mathbb{N}\Delta$ and $w_1 \in N_K(\mathfrak{a}_\mathrm{q})$, then (9.9) holds for λ in a full open subset of Ω as soon as it holds for a fixed λ in the full open subset $\Omega \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY)$ of Ω. We now turn to the proof.

If $s_1 = s$, or more generally, if $s_1 \in sW_Q$, then the conclusion readily follows by the previous remark. By Definition 9.12 we now see that it suffices to prove the lemma for $s_1 = s_\alpha s$, with $\alpha \in \Delta$ such that (9.7) holds. There are two cases to consider, namely that $s^{-1}\alpha|_{\mathfrak{a}^*_{Q\mathrm{q}}}$ equals zero or not. In the first case, $s_1 = ss_{s^{-1}\alpha} \in sW_Q$ and the conclusion is valid. We may thus assume that we are in the second case, i.e., $s_1 = s_\alpha s$ with

$$s^{-1}\alpha|_{\mathfrak{a}^*_{Q\mathrm{q}}} \neq 0. \tag{9.10}$$

We will complete the proof by showing that the following assumption leads to a contradiction.

Assumption: for all $t_1 \in W_Q$, $\eta_1 \in Y$, $\mu_1 \in \mathbb{N}\Delta$ and $w_1 \in N_K(\mathfrak{a}_\mathrm{q})$ there exists no non-empty open subset Ω' of Ω such that (9.9) holds for $\lambda \in \Omega'$.

Let Ξ be the set of elements $(st\eta - \mu)|_{\mathfrak{a}_{\alpha\mathrm{q}}}$ with $t \in W_Q, \eta \in Y, \mu \in \mathbb{N}\Delta$ such that (9.8) holds for λ in a non-empty open subset of Ω, for some $w \in N_K(\mathfrak{a}_\mathrm{q})$. Then Ξ is a non-empty subset of $\mathfrak{a}^*_{\alpha\mathrm{q}\mathbb{C}}$ contained in a set of the form $X - \mathbb{N}\Delta_r(P_\alpha)$, with $X \subset \mathfrak{a}^*_{\alpha\mathrm{q}\mathbb{C}}$ finite. Hence we may select $t \in W_Q, \eta \in Y$ and $\mu \in \mathbb{N}\Delta$ such that $(st\eta - \mu)|_{\mathfrak{a}_{\alpha\mathrm{q}}}$ is $\preceq_{\Delta_r(P_\alpha)}$-maximal in Ξ. According to the first paragraph of the proof, there exists $w \in N_K(\mathfrak{a}_\mathrm{q})$ such that (9.8) is valid for λ in a full open subset Ω_0 of Ω. For $\lambda \in \Omega_0$ we put

$$\xi(\lambda) = [s\lambda + st\eta - \rho - \mu]|_{\mathfrak{a}_{\alpha\mathrm{q}}}.$$

Then by transitivity of asymptotics, see Theorem 3.5, it follows that

$$\xi(\lambda) \in \mathrm{Exp}(P_\alpha, w \,|\, f_\lambda)$$

for $\lambda \in \Omega_0$. In the following we shall investigate the coefficient of the expansion of f_λ along (P_α, w), for $\lambda \in \Omega_0$, given by

$$\varphi_\lambda(m) := q_{\xi(\lambda)}(P_\alpha, w \,|\, f_\lambda, \,\cdot\,, m).$$

Here φ_λ is a non-trivial τ_{P_α}-spherical function on $\mathrm{X}_{\alpha,w,+}$ with values in $P_k(\mathfrak{a}_{\alpha\mathrm{q}})$, for $k = \deg_{\mathrm{a}} f$, see Thm. 3.4 (b).

It follows from (9.10) and the asymptotic globality assumption on f, see Lemma 9.7, that actually φ_λ extends to a smooth function on $\mathrm{X}_{\alpha,w}$, for every λ in an dense open subset Ω_0' of Ω_0. This observation will play a crucial role at a later stage of this proof.

Let

$$\Omega_1 := \Omega_0' \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY) \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P_\alpha, WY).$$

The second and third set in this intersection are full open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, see Lemma 6.3. Hence Ω_1 is a dense open subset of Ω. We claim that for $\lambda \in \Omega_1$ the following holds. If $s' \in W$, $t' \in W_Q$, $\eta' \in Y$, $\mu' \in \mathbb{N}\Delta$ and $w' \in N_K(\mathfrak{a}_\mathrm{q})$ are such that

$$\left\{ \begin{array}{l} s'\lambda + s't'\eta' - \rho - \mu' \in \mathrm{Exp}(P, w' \,|\, f_\lambda) \quad \text{and} \\ \xi(\lambda) \preceq_{\Delta_r(P_\alpha)} (s'\lambda + s't'\eta' - \rho - \mu')|_{\mathfrak{a}_{\alpha\mathrm{q}}}, \end{array} \right. \tag{9.11}$$

then

$$s' \in sW_Q \quad \text{and} \quad (s't'\eta' - \mu')|_{\mathfrak{a}_{\alpha\mathrm{q}}} = (st\eta - \mu)|_{\mathfrak{a}_{\alpha\mathrm{q}}}. \tag{9.12}$$

To prove the claim, let s', t', η', μ', w' satisfy (9.11). Then there exists a $\nu \in \mathbb{N}\Delta(P_\alpha)$ such that $s'\lambda + s't'\eta' - \rho - \mu' - \nu$ and $s\lambda + st\eta - \rho - \mu$ have the same restriction $\xi(\lambda)$ to $\mathfrak{a}_{\alpha\mathrm{q}}$. By the definition of Ω_1 this implies that s' and s define the same class in $W/\sim_{P_\alpha|Q}$, see Lemma 6.2. The latter set equals $W_\alpha\backslash W/W_Q$, by Lemma 6.5, hence s' belongs to $s_\alpha s W_Q = s_1 W_Q$ or to sW_Q. In the first case it follows that $s'\lambda = s_1\lambda$, hence $s_1\lambda + s_1 t''\eta' - \rho - \mu' \in \mathrm{Exp}(P, w' \,|\, f_\lambda)$ for some $t'' \in W_Q$. This assertion then holds for λ in a full open subset of Ω_1, contradicting the above assumption.

It follows that we are in the second case $s' \in sW_Q$, hence $s' = st''$ for some $t'' \in W_Q$. The element $(s't'\eta' - \mu')|_{\mathfrak{a}_{\alpha\mathrm{q}}} = (st''t'\eta' - \mu')|_{\mathfrak{a}_{\alpha\mathrm{q}}}$ therefore belongs to Ξ; from (9.11) it follows that it dominates the maximal element $(st\eta - \mu)|_{\mathfrak{a}_{\alpha\mathrm{q}}}$, hence is equal to that element. This implies (9.12), hence establishes the claim.

It follows from the above claim that, for $\lambda \in \Omega_1$, the exponent $\xi(\lambda)$ is actually a leading exponent of f_λ along (P_α, w). To see this, let $\lambda \in \Omega_1$ and let $\xi \in \mathrm{Exp}(P_\alpha, w \,|\, f_\lambda)$ be an exponent with $\xi(\lambda) \preceq_{\Delta_r(P_\alpha)} \xi$. Then, in view of Theorem 3.5, there exist $s' \in W$, $t' \in W_Q$, $\eta' \in Y$ and $\mu' \in \mathbb{N}\Delta$ such that the element $s'\lambda + s't'\eta' - \rho - \mu'$ restricts to ξ on $\mathfrak{a}_{\alpha\mathrm{q}}$ and belongs to $\mathrm{Exp}(P, w' \,|\, f_\lambda)$ for some $w' \in \mathcal{W}$. It now follows from the claim established above that $\xi = \xi(\lambda)$.

Thus, we see that $\xi(\lambda)$ is a leading exponent indeed. Consequently, by Lemma 5.7 the function φ_λ is $\mathbb{D}(\mathrm{X}_{\alpha,w})$-finite, for every $\lambda \in \Omega_1$. We proceed by investigating the exponents of its expansion.

Select a complete set $\mathcal{W}_{\alpha,w}$ of representatives for $W_\alpha/(W_\alpha \cap W_{K\cap wHw^{-1}})$ in $N_K(\mathfrak{a}_\mathrm{q})$. We put ${}^*P = P \cap M_\alpha$. Then by transitivity of asymptotics, cf. Theorem 3.5, we see that for the set of $({}^*P, u)$-exponents of φ_λ, as $u \in \mathcal{W}_{\alpha,w}$, the following inclusion holds:

$$\mathrm{Exp}({}^*P, u \,|\, \varphi_\lambda) \subset \{\xi|_{{}^*\mathfrak{a}_{\alpha\mathrm{q}}} \mid \xi \in \mathrm{Exp}(P, uw \,|\, f_\lambda) \quad \xi|\mathfrak{a}_{\alpha\mathrm{q}} = \xi(\lambda)|\mathfrak{a}_{\alpha\mathrm{q}}\}.$$

Hence, for $\lambda \in \Omega_1$, every exponent in $\mathrm{Exp}({}^*P, u \,|\, \varphi_\lambda)$ is of the form $(s'\lambda + s't'\eta' - \rho - \mu')|_{{}^*\mathfrak{a}_{\alpha\mathrm{q}}}$ with $s' \in W$, $t' \in W_Q$, $\eta' \in Y$ and $\mu' \in \mathbb{N}\Delta$ satisfying

$$\begin{cases} s'\lambda + s't'\eta' - \rho - \mu' \in \mathrm{Exp}(P, uw \,|\, f_\lambda), \\ [s'\lambda + s't'\eta' - \rho - \mu']|_{\mathfrak{a}_{\alpha\mathrm{q}}} = \xi(\lambda)|_{\mathfrak{a}_{\alpha\mathrm{q}}}. \end{cases}$$

It follows from the claim established above that (9.12) holds.

We have thus shown that for every $\lambda \in \Omega_1$ the exponents in $\mathrm{Exp}({}^*P, u \,|\, \varphi_\lambda)$ are of the form $(s\lambda + st'\eta' - \rho - \mu')|_{{}^*\mathfrak{a}_{\alpha\mathrm{q}}}$ with $t' \in W_Q$, $\eta' \in Y$, $\mu' \in \mathbb{N}\Delta$ satisfying

$$[st'\eta' - \mu']|_{\mathfrak{a}_{\alpha\mathrm{q}}} = [st\eta - \mu]|_{\mathfrak{a}_{\alpha\mathrm{q}}}.$$

From this it follows that the restriction $\mu'|_{\mathfrak{a}_{\alpha\mathrm{q}}}$ of the μ' occurring runs through a finite subset of $\mathbb{N}\Delta_r(P_\alpha) = \mathbb{N}[\Delta \setminus \{\alpha\}]|_{\mathfrak{a}_{\alpha\mathrm{q}}}$, independent of λ. Hence there exists a finite subset $S' \subset \mathbb{N}\Delta$ such that μ' runs through $S' - \mathbb{N}\alpha$. We thus see that there exists a finite subset $S \subset {}^*\mathfrak{a}^*_{\alpha\mathrm{q}\mathbb{C}}$ such that, for every $\lambda \in \Omega_1$,

$$\cup_{u \in \mathcal{W}_{\alpha,w}} \mathrm{Exp}({}^*P, u \,|\, \varphi_\lambda) \subset s\lambda|_{{}^*\mathfrak{a}_{\alpha\mathrm{q}}} + S - \mathbb{N}\alpha. \tag{9.13}$$

From (9.7) and (9.10) it now follows that we may select a non-empty open subset Ω_2 of the dense open subset Ω_1 of Ω such that, for every $\lambda \in \Omega_2$, each $u \in \mathcal{W}_{\alpha,w}$ and all $\xi \in \mathrm{Exp}({}^*P, u \,|\, \varphi_\lambda)$,

$$\langle \mathrm{Re}\,\xi + {}^*\rho\,,\, \alpha\rangle < 0.$$

Since φ_λ is $\mathbb{D}(\mathrm{X}_{\alpha,w})$-finite this implies that φ_λ is square integrable on $\mathrm{X}_{\alpha,w}$, see [3], Thm. 6.4 with $p = 2$; hence φ_λ a Schwartz function for $\lambda \in \Omega_2$, see [3], Thm. 7.3.

On the other hand, from (9.10) it follows that the linear map $\lambda \mapsto s\lambda|_{{}^*\mathfrak{a}_{\alpha\mathrm{q}}}$ is surjective from $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ onto ${}^*\mathfrak{a}^*_{\alpha\mathrm{q}\mathbb{C}}$. Therefore, the set $\{s\lambda|_{{}^*\mathfrak{a}_{\alpha\mathrm{q}}} \mid \lambda \in \Omega_2\}$ has a non-empty interior in ${}^*\mathfrak{a}^*_{\alpha\mathrm{q}\mathbb{C}}$. Combining this observation with (9.13) we infer that there exists a non-empty open subset $\Omega_3 \subset \Omega_2$, such that the sets $\cup_{u \in \mathcal{W}_{\alpha,w}} \mathrm{Exp}({}^*P, u \,|\, \varphi_\lambda)$, for $\lambda \in \Omega_3$, are mutually disjoint. Now these sets are non-empty, since $\varphi_\lambda \neq 0$, for $\lambda \in \Omega_3$. Therefore, the union of these sets, as $\lambda \in \Omega_3$, is uncountable. This contradicts Lemma 5.8, applied to the space $\mathrm{X}_{\alpha,w}$. $\square$

Lemma 9.14 *Assume that $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is Q-distinguished. Then $e \in W(\Omega, s_0)$ for all $s_0 \in W$.*

Proof: Let $k = l(s_0)$ denote the length of s_0, and let $s_0 = s_{\alpha_1} \cdots s_{\alpha_k}$ be a reduced expression for s_0. Put $s_j = s_{\alpha_j} \cdots s_{\alpha_1} s_0 = s_{\alpha_{j+1}} \cdots s_{\alpha_k}$ for $j = 1, \ldots, k$, then $s_k = e$. We claim that (9.7) holds for each pair $(s, \alpha) = (s_{j-1}, \alpha_j)$. Since $l(s_j) = l(s_{j-1}) - 1$, the root $s_{j-1}^{-1}\alpha_j$ must be negative. Hence the restriction of this root to $\mathfrak{a}_{Q\mathrm{q}}$ is zero or belongs to $-\Sigma(Q)$. Now (9.7) follows immediately from Definition 9.9. □

Proof of Theorem 9.10: We prove the result by induction on $|\delta|$. If $\delta = 0$, then for $\lambda \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ the ideal $I_{\delta,\lambda}$ equals $\mathbb{D}(\mathrm{X})$; hence $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}} = 0$ and the result follows.

Let now $|\delta| = 1$, let $f \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ and let (9.4) be fulfilled for all $v \in {}^Q\mathcal{W}$. Assume that $f \neq 0$. We will show that this assumption leads to a contradiction. There exists a finite subset $Y \subset {}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ such that $f \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ and a $\lambda_0 \in \Omega \cap \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*\circ}(P, WY)$ such that $f_{\lambda_0} \neq 0$. Let $\mathcal{W}$ be a complete set of representatives of $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_\mathrm{q})$ containing ${}^Q\mathcal{W}$. Then $\mathrm{Exp}(P, w \,|\, f_{\lambda_0}) \neq \emptyset$ for some $w \in \mathcal{W}$. In view of (7.3) it follows that there exist $s \in W$, $t \in W_Q$, $\eta \in Y$ and $\mu \in \mathbb{N}\Delta$, such that

$$s\lambda + st\eta - \rho - \mu \in \mathrm{Exp}(P, w \,|\, f_\lambda), \tag{9.14}$$

for $\lambda = \lambda_0$. From Lemma 7.9 it follows that (9.14) is valid for λ in a full open subset of Ω. By Lemmas 9.13 and 9.14 this implies that there exist $t_1 \in W_Q$, $\eta_1 \in Y$, $\mu_1 \in \mathbb{N}\Delta$ and $w_1 \in N_K(\mathfrak{a}_\mathrm{q})$, such that $\lambda + t_1\eta_1 - \rho - \mu_1 \in \mathrm{Exp}(P, w_1 \,|\, f_\lambda)$ for λ in a full open subset of Ω. Let $v \in {}^Q\mathcal{W}$ be the representative of $W_Q w_1 W_{K\cap H}$. By Lemma 9.11 it follows that $\lambda - \rho_Q \in \mathrm{Exp}(Q, v \,|\, f_\lambda)$ for λ in a full open subset Ω_0 of Ω. Since $\Omega_0 \cap \Omega'$ is non-empty, we obtain a contradiction with (9.4).

Now suppose that $|\delta| = k > 1$, and assume that the result has already been established for $\delta \in \mathrm{D}_Q$ with $|\delta| < k$. Fix $\nu \in \mathrm{supp}\,(\delta)$ and put $\delta' = \delta - \delta_\nu$. Then $\delta' \in \mathrm{D}_Q$; moreover, $|\delta_\nu| = 1$ and $|\delta'| = k - 1$. Fix any $D \in \mathbb{D}(\mathrm{X})$ and define the family g by (9.3). Then $g \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta')$ by Lemma 9.2. Moreover, it readily follows from Lemma 9.8 that the family g belongs to $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta')_{\mathrm{glob}}$.

For $\lambda \in \Omega$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$ we have

$$\mathrm{Exp}(Q, v \,|\, g_\lambda) \subset \mathrm{Exp}(Q, v \,|\, f_\lambda) - \mathbb{N}\Sigma_r(Q), \tag{9.15}$$

in view of Lemma 4.12 (b). Moreover, by hypothesis we have the following inclusion, for every $\lambda \in \Omega'$,

$$\mathrm{Exp}(Q, v \,|\, f_\lambda) \subset [W(\lambda + Y)|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q - \mathbb{N}\Sigma_r(Q)] \setminus \{\lambda - \rho_Q\}. \tag{9.16}$$

Combining (9.15) and (9.16) we infer that $\mathrm{Exp}(Q, w \,|\, g_\lambda)$ does not contain $\lambda - \rho_Q$ for $\lambda \in \Omega'$ and every $w \in N_K(\mathfrak{a}_\mathrm{q})$. Consequently, the family g satisfies the hypotheses of Theorem 9.10. Since $|\delta'| = k - 1$, it follows from the induction hypothesis that $g = 0$. Since D was arbitrary, we see that f_λ is annihilated by $I_{\delta_\nu,\lambda}$, for every $\lambda \in \Omega$. Hence f belongs to $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta_\nu)_{\mathrm{glob}}$. Since $|\delta_\nu| = 1 < k$, it now follows from the induction hypothesis that $f = 0$. □

The following result is also based on Lemma 9.13.

Corollary 9.15 *Let $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a connected dense open subset, $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset, and $\delta \in \mathrm{D}_Q$. Let $f \in \mathcal{E}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$ and $s_1 \in W$. If*

$$(s_1\lambda + WY - \rho - \mathbb{N}\Delta) \cap \mathrm{Exp}(P, w \,|\, f_\lambda) = \emptyset,$$

for all λ in a non-empty open subset of Ω and for all $w \in N_K(\mathfrak{a}_\mathrm{q})$, then $f = 0$.

Proof: Assume that $f \neq 0$. Then there exists an element $\lambda \in \Omega \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, WY)$ such that $f_\lambda \neq 0$, and then

$$s\lambda + st\eta - \rho - \mu \in \mathrm{Exp}(P, w \,|\, f_\lambda) \tag{9.17}$$

for some $s \in W$, $t \in W_Q$, $\eta \in Y$, $\mu \in \mathbb{N}\Delta$ and $w \in N_K(\mathfrak{a}_\mathrm{q})$. As remarked in the beginning of the proof of Lemma 9.13, (9.17) then holds for all λ in a full open subset of Ω. Hence Lemma 9.13 applies; its final statement contradicts the present assumption for s_1. $\square$

Finally in this section we will show that for a family in $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)$ that allows a smooth extension to X, the hypothesis of asymptotic globality can be left out in the vanishing theorem. Let

$$\mathcal{E}_Q(\mathrm{X} : \tau : \Omega : \delta) = \{f \in \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta) \mid f_\lambda \in C^\infty(\mathrm{X} : \tau), \lambda \in \Omega\}.$$

Corollary 9.16 *Let $Q \in \mathcal{P}_\sigma$ and $\delta \in \mathrm{D}_Q$. Let $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be a Q-distinguished open subset and let $f \in \mathcal{E}_Q(\mathrm{X} : \tau : \Omega : \delta)$. Assume that there exists a non-empty open subset $\Omega' \subset \Omega$ such that, for each $v \in {}^Q\mathcal{W}$,*

$$\lambda - \rho_Q \notin \mathrm{Exp}(Q, v \,|\, f_\lambda), \qquad (\lambda \in \Omega').$$

Then $f = 0$.

Proof: As in the proof of Theorem 9.10 we proceed by induction on $|\delta|$. If $|\delta| = 0$ the result is trivial. If $|\delta| = 1$ it follows from Proposition 8.8 that $\mathcal{E}_Q(\mathrm{X} : \tau : \Omega : \delta) \subset \mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$, and then the result follows directly from Theorem 9.10.

Now suppose that $|\delta| = k > 1$, and assume that the result has already been established for all $\delta \in D_Q$ with $|\delta| < k$. Let δ' and g be as in the proof of Theorem 9.10. Then it is easily seen that $g \in \mathcal{E}_Q(\mathrm{X} : \tau : \Omega : \delta')$.

For the rest of the proof we can now proceed exactly as in the proof of Theorem 9.10. $\square$

10 Laurent functionals

In this section we define Laurent functionals and describe their actions on suitable spaces of meromorphic functions.

Throughout this section, V will be a finite dimensional real linear space, equipped with a (positive definite) inner product $\langle \cdot\, , \,\cdot \rangle$. Its complexification $V_{\mathbb{C}}$ is equipped with the complex bilinear extension of this inner product.

Let X be a (possibly empty) finite set of non-zero elements of V. At this stage we allow proportionality between elements of X. By an X-hyperplane in $V_{\mathbb{C}}$, we mean an affine hyperplane of the form $H = a + \alpha_{\mathbb{C}}^{\perp}$, with $a \in V_{\mathbb{C}}$, $\alpha \in X$. The hyperplane is called real if a can be chosen from V, or, equivalently, if it is the complexification of a real hyperplane from V. A locally finite collection of X-hyperplanes in $V_{\mathbb{C}}$ is called an X-configuration in $V_{\mathbb{C}}$. It is called real if all its hyperplanes are real.

If $a \in V_{\mathbb{C}}$, we denote the collection of X-hyperplanes in $V_{\mathbb{C}}$ through a by $\mathcal{H}(a, X) = \mathcal{H}(V_{\mathbb{C}}, a, X)$. If E is a complete locally convex space, then by $\mathcal{M}(a, X, E) = \mathcal{M}(V_{\mathbb{C}}, a, X, E)$ we denote the ring of germs of E-valued meromorphic functions at a whose singular locus at a is contained in $\mathcal{H}(a, X)$. Here and in the following we will suppress the space E in the notation if $E = \mathbb{C}$. Thus, $\mathcal{M}(a, X) = \mathcal{M}(a, X, \mathbb{C})$.

Let $\mathbb{N}^X$ denote the set of maps $X \to \mathbb{N}$. If $d \in \mathbb{N}^X$, we define the polynomial function $\pi_{a,d} = \pi_{a,X,d} \colon V_{\mathbb{C}} \to \mathbb{C}$ by

$$\pi_{a,d}(z) = \prod_{\xi \in X} \langle \xi\, , \, z - a \rangle^{d(\xi)}, \qquad (z \in V_{\mathbb{C}}). \tag{10.1}$$

If $X = \emptyset$ then $\mathbb{N}^X$ has one element which we agree to denote by 0. We also agree that $\pi_{a,0} = 1$. Let $\mathcal{O}_a(E) = \mathcal{O}_a(V_{\mathbb{C}}, E)$ denote the ring of germs of E-valued holomorphic functions at a. Then

$$\mathcal{M}(a, X, E) = \cup_{d \in \mathbb{N}^X} \ \pi_{a,d}^{-1} \mathcal{O}_a(E).$$

In the following we shall identify $S(V)$ with the algebra of constant coefficient holomorphic differential operators on $V_{\mathbb{C}}$ in the usual way; in particular an element $v \in V$ corresponds to the operator $\varphi \mapsto v\varphi(z) = \frac{d}{d\tau}\big|_{\tau=0} \varphi(z + \tau v)$.

Definition 10.1 (Laurent functional at a point) *An X-Laurent functional at a is a linear functional $\mathcal{L} \colon \mathcal{M}(a, X) \to \mathbb{C}$ such that for every $d \in \mathbb{N}^X$ there exists an element $u_d \in S(V)$ such that*

$$\mathcal{L}\varphi = u_d(\pi_{a,d}\varphi)(a), \tag{10.2}$$

for all $\varphi \in \pi_{a,d}^{-1}\mathcal{O}_a$. The space of all Laurent functionals at a is denoted by $\mathcal{M}(a, X)^*_{\mathrm{laur}} = \mathcal{M}(V_{\mathbb{C}}, a, X)^*_{\mathrm{laur}}$.

Remark 10.2 Obviously, the string $(u_d)_{d\in\mathbb{N}^X}$ of elements from $S(V)$ is uniquely determined by the requirement (10.2). We shall denote it by $u_{\mathcal{L}}$.

If E is a complete locally convex space, then X-Laurent functionals at a may naturally be viewed as linear maps from $\mathcal{M}(a, X, E)$ to E. Indeed, let $\mathcal{L} \in \mathcal{M}(a, X)^*_{\rm laur}$ and let $u_{\mathcal{L}} = (u_d)_{d\in\mathbb{N}^X}$ be the associated string of elements from $S(V)$. If $\varphi \in \pi_{a,d}^{-1}\mathcal{O}_a(E)$ then $\mathcal{L}\varphi$ is given by formula (10.2).

Let $T_a\colon z \mapsto z + a$ denote translation by a in $V_{\mathbb{C}}$. Then T_a maps $\mathcal{H}(0, X)$ bijectively onto $\mathcal{H}(a, X)$. Pull-back under T_a induces an isomorphism of rings $T_a^*\colon \varphi \mapsto \varphi \circ T_a$ from $\mathcal{O}_a$ onto $\mathcal{O}_0$. Therefore, pull-back under T_a also induces an isomorphism of rings $T_a^*\colon \mathcal{M}(a, X) \to \mathcal{M}(0, X)$. By transposition we obtain an isomorphism of linear spaces $T_{a*}\colon \mathcal{M}(0, X)^* \to \mathcal{M}(a, X)^*$. It is readily seen that $T_a^*(\pi_{a,d}) = \pi_{0,d}$ for every $d \in \mathbb{N}^X$. From the definition of Laurent functionals it now follows that T_{a*} maps $\mathcal{M}(0, X)^*_{\rm laur}$ isomorphically onto $\mathcal{M}(a, X)^*_{\rm laur}$. Moreover,

$$u_{T_{a*}\mathcal{L}} = u_{\mathcal{L}}$$

for all $\mathcal{L} \in \mathcal{M}(0, X)^*$.

Let X' be another finite collection of non-zero elements of V. We say that X and X' are proportional if $\mathcal{H}(0, X) = \mathcal{H}(0, X')$.

Lemma 10.3 *Let X, X' be proportional finite subsets of $V \setminus \{0\}$ and let $a \in V_{\mathbb{C}}$. Then $\mathcal{M}(a, X) = \mathcal{M}(a, X')$ and $\mathcal{M}(a, X)^*_{\rm laur} = \mathcal{M}(a, X')^*_{\rm laur}$.*

Proof: It is obvious that $\mathcal{M}(a, X) = \mathcal{M}(a, X')$. Let $\mathcal{L} \in \mathcal{M}(a, X)^* = \mathcal{M}(a, X')^*$, and assume that $\mathcal{L} \in \mathcal{M}(a, X')^*_{\rm laur}$. Let $(u_{d'})_{d'\in\mathbb{N}^{X'}}$ be the associated string. Let $d \in \mathbb{N}^X$. Then, by proportionality, there exists $d' \in \mathbb{N}^{X'}$ and $c \in \mathbb{R} \setminus \{0\}$ such that $\pi_{a,X,d} = c\pi_{a,X',d'}$. Let $u_d = c^{-1}u_{d'}$, then (10.2) follows immediately. This shows that $\mathcal{L} \in \mathcal{M}(a, X)^*_{\rm laur}$ and establishes the inclusion $\mathcal{M}(a, X')^*_{\rm laur} \subset \mathcal{M}(a, X)^*_{\rm laur}$. The converse inclusion is proved similarly. $\square$

Following the method of [10], Sect. 1.3, we shall now give a description of the space of strings $u_{\mathcal{L}}$, as $\mathcal{L} \in \mathcal{M}(a, X)^*_{\rm laur}$.

Put $\varpi_d\colon = \pi_{0,d}$ and equip the space $\mathbb{N}^X$ with the partial ordering $\preceq$ defined by $d' \preceq d$ if and only if $d'(\xi) \leq d(\xi)$ for every $\xi \in X$. If $d' \preceq d$ then we define $d - d'$ componentwise as suggested by the notation. In [10], Sect. 1.3, we defined the linear space $S_{\leftarrow}(V, X)$ as follows. Let $d, d' \in \mathbb{N}^X$ with $d' \preceq d$. If $u \in S(V)$, then by the Leibniz rule there exists a unique $u' \in S(V)$ such that

$$u(\varpi_{d-d'}\varphi)(0) = u'(\varphi)(0), \qquad (\varphi \in \mathcal{O}_0).$$

We denote the element u' by $j_{d',d}(u)$. The map $j_{d',d}\colon S(V) \to S(V)$ thus defined is linear. Note that it only depends on $d - d'$; note also that, for $d, d', d'' \in \mathbb{N}^X$ with $d'' \preceq d' \preceq d$,

$$j_{d'',d'} \circ j_{d',d} = j_{d'',d}.$$

We now define $S_{\leftarrow}(V, X)$ as the linear space of strings $(u_d)_{d\in\mathbb{N}^X}$ in $S(V)$ such that $j_{d',d}(u_d) = u_{d'}$ for all $d, d \in \mathbb{N}^X$ with $d' \preceq d$. Thus, this space is the

projective limit:
$$S_{\leftarrow}(V,X) = \lim_{\leftarrow}(S(V), j_{\cdot}).$$

The natural map $S_{\leftarrow}(V,X) \to S(V)$ that maps a string to its d-component is denoted by j_d.

Lemma 10.4 *The map $\mathcal{L} \mapsto u_{\mathcal{L}}$ is a linear isomorphism from $\mathcal{M}(a,X)^*_{\mathrm{laur}}$ onto $S_{\leftarrow}(V,X)$.*

Proof: See [11], Appendix B, Lemma B.2. $\square$

Lemma 10.5 *Let $a \in V_{\mathbb{C}}$, $d \in \mathbb{N}^X$ and $u \in S(V)$. Then there exists a Laurent functional $\mathcal{L} \in \mathcal{M}(a,X)^*_{\mathrm{laur}}$ such that $(u_{\mathcal{L}})_d = u$.*

Proof: See [10], Lemma 1.7. $\square$

Remark 10.6 In particular, it follows that for each $a \in V_{\mathbb{C}}$ there exists a Laurent functional $\mathcal{L} \in \mathcal{M}(a,X)^*_{\mathrm{laur}}$ such that $\mathcal{L}\varphi = \varphi(a)$ for all $\varphi \in \mathcal{O}_a$. Note however, that this functional is not unique, unless $X = \emptyset$.

Lemma 10.7 *Let $\mathcal{M}(a,X)^{*\mathcal{O}}_{\mathrm{laur}}$ denote the annihilator of $\mathcal{O}_a$ in $\mathcal{M}(a,X)^*_{\mathrm{laur}}$. Then all functions φ in $\mathcal{M}(a,X)$, that are annihilated by $\mathcal{M}(a,X)^{*\mathcal{O}}_{\mathrm{laur}}$, belong to $\mathcal{O}_a$.*

Proof: We may assume that $a = 0$. Let $\varphi \in \mathcal{M}(0,X)$ and assume that $\varphi \notin \mathcal{O}_0$. Then there exist elements $d, d' \in \mathbb{N}^X$ and $\xi \in X$ such that $\pi_{0,d'} = \xi\pi_{0,d}$ and $\pi_{0,d'}\varphi \in \mathcal{O}_0$ but $\pi_{0,d}\varphi \notin \mathcal{O}_0$. Here we have written ξ also for the function $z \mapsto \langle \xi\,,\, z\rangle$ on $V_{\mathbb{C}}$. Since $\pi_{0,d'}\varphi$ is not divisible by ξ, its restriction to $\xi^{\perp} = \xi^{-1}(0)$ does not vanish. Hence there exists $u \in S(\xi^{\perp})$ such that $u(\pi_{0,d'}\varphi)(0) \neq 0$. By Lemma 10.5 there exists an element $\mathcal{L} \in \mathcal{M}(a,X)^*_{\mathrm{laur}}$ such that the d' term of $u_{\mathcal{L}}$ is u. Then $\mathcal{L}\varphi = u(\pi_{0,d'}\varphi)(0) \neq 0$. However, for each $\psi \in \mathcal{O}_0$ we have $\mathcal{L}\psi = u(\pi_{0,d'}\psi)(0) = [\xi u(\pi_{0,d}\psi)](0) = 0$. Hence $\mathcal{L} \in \mathcal{M}(a,X)^{*\mathcal{O}}_{\mathrm{laur}}$. $\square$

We extend the notion of a Laurent functional as follows. The disjoint union of the spaces $\mathcal{M}(a,X)^*_{\mathrm{laur}}$ as $a \in V_{\mathbb{C}}$ is denoted by $\mathcal{M}(*,X)^*_{\mathrm{laur}} = \mathcal{M}(V_{\mathbb{C}},*,X)^*_{\mathrm{laur}}$. By a section of $\mathcal{M}(*,X)^*_{\mathrm{laur}}$ we mean a map $\mathcal{L}\colon V_{\mathbb{C}} \to \mathcal{M}(*,X)^*_{\mathrm{laur}}$ with $\mathcal{L}_a \in \mathcal{M}(a,X)^*_{\mathrm{laur}}$ for all $a \in V_{\mathbb{C}}$. The closure of the set $\{a \in V_{\mathbb{C}} \mid \mathcal{L}_a \neq 0\}$ is called the support of $\mathcal{L}$ and denoted by $\mathrm{supp}\,(\mathcal{L})$.

Definition 10.8 (Laurent functional) *An X-Laurent functional on $V_{\mathbb{C}}$ is a finitely supported section of $\mathcal{M}(*,X)^*_{\mathrm{laur}}$. The set of X-Laurent functionals is denoted by $\mathcal{M}(V_{\mathbb{C}},X)^*_{\mathrm{laur}}$ and equipped with the obvious structure of a linear space.*

Is S is a subset of $V_{\mathbb{C}}$, we define the space $\mathcal{M}(S, X)^*_{\text{laur}} = \mathcal{M}(V_{\mathbb{C}}, S, X)^*_{\text{laur}}$ by

$$\mathcal{M}(S, X)^*_{\text{laur}} = \{\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}} \mid \operatorname{supp} \mathcal{L} \subset S\}$$

and call this the space of X-Laurent functionals on $V_{\mathbb{C}}$ supported in S.

Remark 10.9 Note that, for $a \in V_{\mathbb{C}}$, the map $\mathcal{M}(\{a\}, X)^*_{\text{laur}} \to \mathcal{M}(a, X)^*_{\text{laur}}$, defined by $\mathcal{L} \mapsto \mathcal{L}_a$, is a linear isomorphism. Accordingly we shall view $\mathcal{M}(a, X)^*_{\text{laur}}$ as a linear subspace of $\mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}}$. In this way $\mathcal{M}(S, X)^*_{\text{laur}}$ becomes identified with the algebraic direct sum of the linear spaces $\mathcal{M}(a, X)^*_{\text{laur}}$, as $a \in S$, for S any subset of $V_{\mathbb{C}}$. Accordingly, if $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}}$, then $\mathcal{L}_a \in \mathcal{M}(a, X)^*_{\text{laur}} \subset \mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}}$ for $a \in V_{\mathbb{C}}$, and

$$\mathcal{L} = \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a.$$

Lemma 10.10 *Let X and X' be proportional finite subsets of $V \setminus \{0\}$. Then*

$$\mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}} = \mathcal{M}(V_{\mathbb{C}}, X')^*_{\text{laur}}.$$

Proof: This is an immediate consequence of Lemma 10.3 and the above definition. $\square$

We proceed by discussing the action of a Laurent functional on meromorphic functions. Let E be a complete locally convex space and $\Omega \subset V_{\mathbb{C}}$ an open subset. If $a \in \Omega$, then by $\mathcal{M}(\Omega, a, X, E)$ we denote the space of meromorphic functions $\varphi\colon \Omega \to E$ whose germ φ_a at a belongs to $\mathcal{M}(a, X, E)$. If $S \subset \Omega$, we define

$$\mathcal{M}(\Omega, S, X, E) := \cap_{a \in S} \mathcal{M}(\Omega, a, X, E).$$

Finally, we write $\mathcal{M}(\Omega, X, E)$ for $\mathcal{M}(\Omega, \Omega, X, E)$. In particular, $\mathcal{M}(V_{\mathbb{C}}, X, E)$ denotes the space of functions $\varphi \in \mathcal{M}(V_{\mathbb{C}}, E)$ with singular locus $\operatorname{sing}(\varphi)$ contained in an X-configuration.

There is a natural pairing $\mathcal{M}(S, X)^*_{\text{laur}} \times \mathcal{M}(\Omega, S, X, E) \to E$, given by

$$\mathcal{L}\varphi = \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a \varphi_a. \tag{10.3}$$

Lemma 10.11 *Let $S \subset V_{\mathbb{C}}$ be arbitrary, and let Ω be an open subset of $V_{\mathbb{C}}$ containing S. Then the pairing given by (10.3) for $E = \mathbb{C}$ induces a linear embedding*

$$\mathcal{M}(S, X)^*_{\text{laur}} \hookrightarrow \mathcal{M}(\Omega, S, X)^*.$$

Proof: Let $\mathcal{L} \in \mathcal{M}(S, X)^*_{\mathrm{laur}}$ and assume that $\mathcal{L} = 0$ on $\mathcal{M}(\Omega, S, X)$. We may assume that $S = \operatorname{supp} \mathcal{L}$. For every $a \in S$ we write $u^a = (u^a_d)_{d \in \mathbb{N}^X}$ for the string determined by $\mathcal{L}_a$.

Select $b \in S$. Then it suffices to prove that $\mathcal{L}_b = 0$. Fix $d \in \mathbb{N}^X$ and $\phi \in \mathcal{O}_b$. Then it suffices to show that $u^b_d(\phi)(b) = 0$.

For every $a \in S \setminus \{b\}$ we may select $d(a) \in \mathbb{N}^X$ such that $\pi_{a,d(a)}\pi^{-1}_{b,d}$ is holomorphic at a. Moreover, we put $d(b) = d$. For $a \in S$ there exists a unique $v_a \in S(V)$ such that for all $f \in \mathcal{O}_a$ we have

$$v_a(f)(a) = u^a_{d(a)}(\pi_{a,d(a)}\pi^{-1}_{b,d} f)(a).$$

We note that $v_b = u^b_d$. We may now apply the lemma below, with $E_a = \mathbb{C}v_a$, for $a \in S$, and, finally with $\xi_a = 0$ if $a \neq b$ and with ξ_b defined by $\xi_b(v_b) = v_b(\phi)(b)$. Hence there exists a polynomial function ψ on $V_\mathbb{C}$ such that $v_a(\psi)(a) = 0$ for all $a \in S \setminus \{b\}$, and such that $v_b(\psi)(b) = v_b(\phi)(b)$.

Define $\varphi = \pi^{-1}_{b,d}\psi$. Then $\varphi \in \mathcal{M}(\Omega, S, X)$. Hence $\mathcal{L}\varphi = 0$. On the other hand,

$$\begin{aligned}
\mathcal{L}\varphi = \sum_{a \in S} \mathcal{L}_a \varphi_a &= \sum_{a \in S} \mathcal{L}_a(\pi^{-1}_{a,d(a)}\pi_{a,d(a)}\pi^{-1}_{b,d}\psi) \\
&= \sum_{a \in S} u^a_{d(a)}(\pi_{a,d(a)}\pi^{-1}_{b,d}\psi)(a) = \sum_{a \in S} v_a(\psi)(a) = v_b(\psi)(b) = u^b_d(\phi)(b).
\end{aligned}$$

It follows that $u^b_d(\phi)(b) = 0$. □

Lemma 10.12 *Let $S \subset V_\mathbb{C}$ be a finite set. Suppose that for every $a \in S$ a finite dimensional complex linear subspace $E_a \subset S(V)$ together with a complex linear functional $\xi_a \in E^*_a$ is given. Then there exists a polynomial function ψ on $V_\mathbb{C}$ such that $u\psi(a) = \xi_a(u)$ for every $a \in S$ and all $u \in E_a$.*

Proof: This result is well known. □

We proceed by discussing the push-forward of a Laurent functional by an injective linear mapping. Let V_0 be a real linear space and $\iota: V_0 \to V$ an injective linear map. We assume that no element of X is orthogonal to $\iota(V_0)$. We equip V_0 with the pull-back of the inner product of V under ι and denote the corresponding transpose of ι by p. Then $X_0 := p(X)$ consists of non-zero elements. We denote the complex linear extensions of ι and p by the same symbols. Then, if $H \subset V_\mathbb{C}$ is an X-hyperplane, its preimage $\iota^{-1}(H)$ is an X_0-hyperplane of $V_{0\mathbb{C}}$.

Let $a_0 \in V_{0\mathbb{C}}$ and put $a = \iota(a_0)$. Then pull-back by ι induces a natural algebra homomorphism $\iota^*: \mathcal{O}_a(V_\mathbb{C}) \to \mathcal{O}_{a_0}(V_{0\mathbb{C}})$. On the other hand, pull-back by p induces a natural algebra homomorphism $p^*: \mathcal{O}_{a_0}(V_{0\mathbb{C}}) \to \mathcal{O}_a(V_\mathbb{C})$. From $p \circ \iota = I_{V_0}$ it follows that $\iota^* \circ p^* = I$ on $\mathcal{O}_{a_0}(V_{0\mathbb{C}})$, hence ι^* is surjective.

If $d: X \to \mathbb{N}$ is a map, then we write $p_*(d)$ for the map $X_0 \to \mathbb{N}$ defined by

$$p_*(d)(\xi_0) = \sum_{\xi \in X, p(\xi) = \xi_0} d(\xi).$$

One readily verifies that for every $d: X \to \mathbb{N}$ we have

$$\iota^*(\pi_{a,X,d}) = \pi_{a_0,X_0,p_*(d)}. \tag{10.4}$$

Let E be a complete locally convex space. Then it follows that pull-back by ι induces a linear map

$$\iota^*: \mathcal{M}(V_\mathbb{C}, a, X, E) \to \mathcal{M}(V_{0\mathbb{C}}, a_0, X_0, E). \tag{10.5}$$

Lemma 10.13 *The linear map ι^* in (10.5) is surjective.*

Proof: Let $d_0: X_0 \to \mathbb{N}$ be a map. Then one readily checks that there exists a map $d: X \to \mathbb{N}$ such that $d_0 = p_*(d)$. From this it follows that

$$\pi^{-1}_{a_0,X_0,d_0}\mathcal{O}_{a_0}(V_{0\mathbb{C}}, E) = \iota^*(\pi^{-1}_{a,X,d})\, \iota^* p^*(\mathcal{O}_{a_0}(V_{0\mathbb{C}}, E)) \subset \iota^*(\pi^{-1}_{a,X,d}\mathcal{O}_a(V_\mathbb{C}, E)).$$

where the first equality follows from (10.4). $\square$

The pull-back map ι^* in (10.5) with $E = \mathbb{C}$ has a transpose $\iota_*: \mathcal{M}(V_{0\mathbb{C}}, a_0, X_0)^* \to \mathcal{M}(V_\mathbb{C}, a, X)^*$ which is injective by Lemma 10.13.

Lemma 10.14 *The map ι_* maps $\mathcal{M}(V_{0\mathbb{C}}, a_0, X_0)^*_{\mathrm{laur}}$ injectively into $\mathcal{M}(V_\mathbb{C}, a, X)^*_{\mathrm{laur}}$.*

Proof: Let $\mathcal{L} \in \mathcal{M}(V_{0\mathbb{C}}, a_0, X_0)^*_{\mathrm{laur}}$. Then it suffices to show that $\iota_*\mathcal{L}$ belongs to the space $\mathcal{M}(V_\mathbb{C}, a, X)^*_{\mathrm{laur}}$.

We first note that $\iota: V_0 \to V$ has a unique extension to an algebra homomorphism $\iota_*: S(V_0) \to S(V)$. One readily verifies that $u[\iota^*(\varphi)] = \iota^*(\iota_*(u)\varphi)$ for every $\varphi \in \mathcal{O}_a(V_\mathbb{C})$ and all $u \in S(V_0)$. Let d be a map $X \to \mathbb{N}$. Then there exists a $u_d \in S(V_0)$ such that $\mathcal{L} = \mathrm{ev}_{a_0} \circ u_d \circ \pi_{a_0,X_0,p_*(d)}$ on $\pi^{-1}_{a_0,X_0,p_*(d)}\mathcal{O}_{a_0}(V_{0\mathbb{C}})$; here ev_{a_0} denotes evaluation at the point a_0. Put $v_d = \iota_*(u_d)$. Then, for $\varphi \in \mathcal{O}_a(V_\mathbb{C})$,

$$\iota_*(\mathcal{L})[\pi^{-1}_{a,X,d}\varphi] = \mathcal{L}[\iota^*(\pi_{a,X,d})^{-1}\iota^*\varphi)] = \mathcal{L}[\pi^{-1}_{a_0,X_0,p_*d}\iota^*\varphi] = \iota^*(v_d\varphi)(a_0) = v_d\varphi(a).$$

Hence $\iota_*(\mathcal{L}) = \mathrm{ev}_a \circ v_d \circ \pi_{a,X,d}$ on $\pi^{-1}_{a,X,d}\mathcal{O}_a(V_\mathbb{C})$ and we see that $\iota_*(\mathcal{L}) \in \mathcal{M}(V_\mathbb{C}, a, X)^*_{\mathrm{laur}}$. $\square$

There exists a unique linear map $\iota_*: \mathcal{M}(V_{0\mathbb{C}}, X_0)^*_{\mathrm{laur}} \to \mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}}$ that restricts to the map ι_* of Lemma 10.14 for every $a_0 \in V_{0\mathbb{C}}$, see Remark 10.9. Clearly, $\mathrm{supp}\,(\iota_*\mathcal{L}) = \iota(\mathrm{supp}\,(\mathcal{L}))$, for every $\mathcal{L} \in \mathcal{M}(V_{0\mathbb{C}}, X_0)^*_{\mathrm{laur}}$.

On the other hand, if E is a complete locally convex space, $\Omega \subset V_\mathbb{C}$ open subset and $S \subset \iota^{-1}(\Omega)$ a subset, then pull-back by ι induces a natural map $\iota^*: \mathcal{M}(\Omega, \iota(S), X, E) \to \mathcal{M}(\iota^{-1}(\Omega), S, X_0, E)$. Moreover, if $\mathcal{L} \in \mathcal{M}(V_{0\mathbb{C}}, S, X_0)^*_{\mathrm{laur}}$ and $\varphi \in \mathcal{M}(\Omega, \iota(S), X, E)$, then

$$\iota_*(\mathcal{L})\varphi = \mathcal{L}[\iota^*\varphi]. \tag{10.6}$$

We end this section with a discussion of the multiplication by a meromorphic function and the application of a differential operator to a Laurent functional.

First, assume that $a \in V_{\mathbb{C}}$ and that $\psi \in \mathcal{M}(a, X)$. Then multiplication by ψ induces a linear endomorphism of $\mathcal{M}(a, X)$, which we denote by m_ψ. The transpose of this linear endomorphism is denoted by $m_\psi^*: \mathcal{M}(a, X)^* \to \mathcal{M}(a, X)^*$. It readily follows from the definition of X-Laurent functionals at a that m_ψ^* leaves the space $\mathcal{M}(a, X)^*_{\text{laur}}$ of those functionals invariant.

Let now $S \subset V_{\mathbb{C}}$ be a finite subset, let $\Omega \subset V_{\mathbb{C}}$ be an open subset containing S and let $\psi \in \mathcal{M}(\Omega, S, X)$. If $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, S, X)^*_{\text{laur}}$, we define the Laurent functional $m_\psi^*(\mathcal{L}) \in \mathcal{M}(V_{\mathbb{C}}, S, X)^*_{\text{laur}}$ by

$$m_\psi^*(\mathcal{L}) = \sum_{a \in S} m_{\psi_a}^*(\mathcal{L}_a).$$

On the other hand, multiplication by ψ induces a linear endomorphism of $\mathcal{M}(\Omega, S, X)$, and it is immediate from the definitions that

$$m_\psi^*(\mathcal{L})(\varphi) = \mathcal{L}(\psi\varphi) \tag{10.7}$$

for $\varphi \in \mathcal{M}(\Omega, S, X)$.

Lemma 10.15 *Let $v \in S(V)$, then $v\varphi \in \mathcal{M}(a, X)$ for all $\varphi \in \mathcal{M}(a, X)$, and the transpose ∂_v^* of the endomorphism $\partial_v: \varphi \mapsto v\varphi$ of $\mathcal{M}(a, X)$ leaves $\mathcal{M}(a, X)^*_{\text{laur}}$ invariant.*

Proof: We may assume $v \in V$. Let $d \in \mathbb{N}^X$ and define $d' \in \mathbb{N}^X$ by $d'(\xi) = d(\xi) + 1$ for all $\xi \in X$. Then $\pi_{a,d}$ divides $v(\pi_{a,d'})$, and hence

$$\pi_{a,d'} v\varphi = v(\pi_{a,d'}\varphi) - v(\pi_{a,d'})\varphi \in \mathcal{O}_a$$

for all $\varphi \in \pi_{a,d}^{-1}\mathcal{O}_a$. Thus $\partial_v\varphi = v\varphi \in \pi_{a,d'}^{-1}\mathcal{O}_a$ for $\varphi \in \pi_{a,d}^{-1}\mathcal{O}_a$.

Let now $\mathcal{L} \in \mathcal{M}(a, X)^*_{\text{laur}}$, and let $u = u_{\mathcal{L}} \in S_{\leftarrow}(V, X)$. Then for d, d' and φ as above

$$\partial_v^*\mathcal{L}(\varphi) = \mathcal{L}(v\varphi) = u_{d'}(\pi_{a,d'}v\varphi)(a) = u_{d'}v(\pi_{a,d'}\varphi)(a) - u_{d'}(v(\pi_{a,d'})\varphi)(a).$$

Each term on the right hand side of this equation has the form $u'(p\varphi)(a)$ with $u' \in S(V)$ and p a polynomial which is divisible by $\pi_{a,d}$. Hence, by the Leibniz rule, $\partial_v^*\mathcal{L}(f)$ has the required form $u''(\pi_{a,d}\varphi)(a)$, where $u'' \in S(V)$. $\square$

For $\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}}$ and $v \in S(V)$ we now define $\partial_v^*\mathcal{L} \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\text{laur}}$ by

$$\partial_v^*\mathcal{L} = \sum_{a \in \operatorname{supp}\mathcal{L}} \partial_v^*\mathcal{L}_a.$$

It is immediately seen that $\partial_v^*\mathcal{L}(\varphi) = \mathcal{L}(\partial_v\varphi)$ for each $\varphi \in \mathcal{M}(\Omega, \operatorname{supp}\mathcal{L}, X)$, where Ω is an arbitrary open neighborhood of $\operatorname{supp}\mathcal{L}$.

11 Laurent operators

In this section we discuss Laurent operators, originally introduced in [10], Section 5, in the slightly different context of meromorphic functions with values in a complete locally convex space, whose singular locus is contained in an X-configuration.

Let V and X be as in the previous section, let $\mathcal{H}$ be an X-configuration and let E be a complete locally convex space.

We define $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ to be the space of meromorphic functions $\varphi: V_{\mathbb{C}} \to E$ whose singular locus is contained in $\cup\mathcal{H}$. If $\mathcal{H}$ is real, we put $\mathcal{H}_V = \{H \cap V \mid H \in \mathcal{H}\}$. Then $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}) = \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, \mathbb{C})$ equals the space $\mathcal{M}(V, \mathcal{H}_V)$ introduced in [10].

It is convenient to select a minimal subset X^0 of X that is proportional to X. Then for every X-hyperplane $H \subset V_{\mathbb{C}}$ there exists a unique $\alpha_H \in X^0$ and a unique first order polynomial l_H of the form $z \mapsto \langle \alpha_H\,,\, z\rangle - c$, with $c \in \mathbb{C}$, such that $H = l_H^{-1}(0)$. Note that a different choice of X^0 causes only a change of l_H by a non-zero factor.

Let $\mathbb{N}^{\mathcal{H}}$ denote the collection of maps $\mathcal{H} \to \mathbb{N}$.

Remark 11.1 If $d \in \mathbb{N}^{\mathcal{H}}$, then for convenience we agree to write $d(H) = 0$ for any X-hyperplane H not contained in $\mathcal{H}$.

If $\omega \subset V_{\mathbb{C}}$ is a bounded subset and $d \in \mathbb{N}^{\mathcal{H}}$ we define the polynomial function $\pi_{\omega,d}: V_{\mathbb{C}} \to \mathbb{C}$ by

$$\pi_{\omega,d} = \prod_{\substack{H \in \mathcal{H} \\ H \cap \omega \neq \emptyset}} l_H^{d(H)} \tag{11.1}$$

Note that a change of X^0 only causes this polynomial to be multiplied by a positive factor. Let $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ be the collection of meromorphic functions $\varphi \in \mathcal{M}(V_{\mathbb{C}}, E)$ such that $\pi_{\omega,d}\varphi \in \mathcal{O}(\omega, E)$ for every bounded open subset $\omega \subset V_{\mathbb{C}}$. We equip the space $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ with the weakest locally convex topology such that for every bounded open subset $\omega \subset V_{\mathbb{C}}$ the map map $\varphi \mapsto \pi_{\omega,d}\varphi$ is continuous into $\mathcal{O}(\omega, E)$. This topology is complete; moreover, it is Fréchet if E is Fréchet.

We now note that

$$\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E) = \cup_{d \in \mathbb{N}^{\mathcal{H}}} \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E). \tag{11.2}$$

We equip $\mathbb{N}^{\mathcal{H}}$ with the partial ordering $\preceq$ defined by $d' \preceq d$ if and only if $d'(H) \leq d(H)$ for all $H \in \mathcal{H}$. If d, d' are elements of $\mathbb{N}^{\mathcal{H}}$ with $d' \preceq d$ then $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d', E) \subset \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ and the inclusion map $i_{d',d}$ is continuous. Thus, the inclusion maps form a directed family and from (11.2) we see that the space $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ may be viewed as the direct limit of the spaces $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$. Accordingly we equip $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ with the direct limit locally convex topology.

By an X-subspace of $V_{\mathbb{C}}$ we mean any non-empty intersection of X-hyperplanes; we agree that $V_{\mathbb{C}}$ itself is also an X-subspace. We denote the set of such affine

subspaces by $\mathcal{A} = \mathcal{A}(V_{\mathbb{C}}, X)$. For $L \in \mathcal{A}$ there exists a unique real linear subspace $V_L \subset V$ such that $L = a + V_{L\mathbb{C}}$ for some $a \in V_{\mathbb{C}}$. The intersection $V_{L\mathbb{C}}^{\perp} \cap L$ consists of a single point, called the central point of L; it is denoted by $c(L)$. The space L is said to be real if $c(L) \in V$; this means precisely that L is the complexification of an affine subspace of V. Translation by $c(L)$ induces an affine isomorphism from $V_{L\mathbb{C}}$ onto L. Via this isomorphism we equip L with the structure of a complex linear space together with a real form that is equipped with an inner product.

If $L \in \mathcal{A}$, the collection of X-hyperplanes containing L is finite; we denote this collection by $\mathcal{H}(L, X)$. Moreover, we put $X(L) := X \cap V_L^{\perp}$ and $X^0(L) := X^0 \cap V_L^{\perp}$. From the definition of X^0 it follows that the map $H \mapsto \alpha_H$ is a bijection from $\mathcal{H}(L, X)$ onto $X^0(L)$. Accordingly we shall identify the sets $\mathbb{N}^{\mathcal{H}(L,X)}$ and $\mathbb{N}^{X^0(L)}$. If $\mathcal{H}$ is any X-configuration and $d \in \mathbb{N}^{\mathcal{H}}$, we define the polynomial function $q_{L,d}$ by

$$q_{L,d} := \prod_{H \in \mathcal{H}(L,X)} l_H^{d(H)},$$

see also Remark 11.1. Let X_r be the orthogonal projection of $X \setminus X(L)$ onto V_L; then X_r is a finite set of non-zero elements. Its image in L under translation by $c(L)$ is denoted by X_L. If $\mathcal{H}$ is an X-configuration in $V_{\mathbb{C}}$, then the collection

$$\mathcal{H}_L := \{H \cap L \mid H \in \mathcal{H},\ \emptyset \subsetneqq H \cap L \subsetneqq L\}$$

is an X_L-configuration in L; here L is viewed as a complex linear space in the way described above.

We now assume that $L \in \mathcal{A}$ and that $\mathcal{H}$ is an X-configuration in $V_{\mathbb{C}}$. In accordance with [10], Sect. 1.3, a linear map $R: \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ is called a Laurent operator if for every $d \in \mathcal{H}^{\mathbb{N}}$ there exists an element $u_d \in S(V_L^{\perp})$ such that

$$R\varphi = u_d(q_{L,d}\varphi)|_L \quad \text{for all} \quad \varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d). \tag{11.3}$$

The space of such Laurent operators is denoted by $\mathrm{Laur}\,(V_{\mathbb{C}}, L, \mathcal{H})$.

Assume now in addition that $\mathcal{H}$ contains $\mathcal{H}(L, X)$. Then as in loc. cit. it is seen that, for $R \in \mathrm{Laur}\,(V_{\mathbb{C}}, L, \mathcal{H})$ and $d \in \mathbb{N}^{\mathcal{H}}$, the element $u_d \in S(V_L^{\perp})$ such that (11.3) holds, is uniquely determined. Moreover, it only depends on the restriction of d to $\mathcal{H}(L, X)$, and the associated string $u_R := (u_d \mid d \in \mathbb{N}^{\mathcal{H}(L,X)})$ belongs to $S_{\leftarrow}(V_L^{\perp}, X^0(L))$. As in to [10], Lemma 1.5, the map $R \mapsto u_R$ defines a linear isomorphism

$$\mathrm{Laur}\,(V_{\mathbb{C}}, L, \mathcal{H}) \simeq S_{\leftarrow}(V_L^{\perp}, X^0(L)). \tag{11.4}$$

If E is a complete locally convex space, and $R \in \mathrm{Laur}\,(V_{\mathbb{C}}, L, \mathcal{H})$ a Laurent operator, we may define a linear operator R_E from $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ to $\mathcal{M}(L, \mathcal{H}_L, E)$ by the formula (11.3), for $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ and with u_d equal to the d-component of u_R. We shall often denote R_E by R as well.

Remark 11.2 Here we note that the algebraic tensor product $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}) \otimes E$ naturally embeds onto a subspace of $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ which is dense. Thus, R_E is the unique continuous linear extension of $R \otimes I_E$. However, we shall not need this.

Lemma 11.3 *Let $L \in \mathcal{A}$ and let $\mathcal{H}$ be an X-configuration in $V_{\mathbb{C}}$ containing $\mathcal{H}(L, X)$. Let $R \in \mathrm{Laur}\,(V_{\mathbb{C}}, L, \mathcal{H})$. Then for every $d \in \mathbb{N}^{\mathcal{H}}$ there exists a $d' \in \mathbb{N}^{\mathcal{H}_L}$ with the following property. For every complete locally convex space E the operator R_E maps $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ continuously into the space $\mathcal{M}(L, \mathcal{H}_L, d', E)$.*

Proof: This is proved in a similar fashion as [10], Lemma 1.10. □

We shall now relate Laurent operators to the Laurent functionals introduced in the previous section. Let X_r^0 be a minimal subset of X_r subject to the condition that it be proportional to X_r. Let X_L^0 be its image in L under translation by $c(L)$. Thus, with respect to the linear structure of L, the set X_L^0 is an analogue for the pair (L, X_L) of the set X^0 for the pair (V, X).

Lemma 11.4 *Let $L \in \mathcal{A}$ and let $\mathcal{H}$ be an X-configuration in $V_{\mathbb{C}}$ containing $\mathcal{H}(L, X)$. Let E be a complete locally convex space.*

(a) *If $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$, then for $w \in L \setminus \cup\mathcal{H}_L$ the function $z \mapsto \varphi(w + z)$ is meromorphic on $V_{L\mathbb{C}}^{\perp}$, with a germ at 0 that belongs to $\mathcal{M}(V_{L\mathbb{C}}^{\perp}, 0, X(L), E)$.*

(b) *If $\mathcal{L} \in \mathcal{M}(V_{L\mathbb{C}}^{\perp}, 0, X(L))^*_{\mathrm{laur}}$ is an $X(L)$-Laurent functional in $V_{L\mathbb{C}}^{\perp}$, supported at the origin, then for $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ the function*

$$\mathcal{L}_* \varphi \colon w \mapsto \mathcal{L}(\varphi(w + \cdot\,)) \qquad (11.5)$$

belongs to the space $\mathcal{M}(L, \mathcal{H}_L, E)$. The operator $\mathcal{L}_* \colon \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$, defined by (11.5) for $E = \mathbb{C}$, is a Laurent operator.

(c) *The map $\mathcal{L} \mapsto \mathcal{L}_*$, defined by (11.5) for $E = \mathbb{C}$, is an isomorphism from the space $\mathcal{M}(V_{L\mathbb{C}}^{\perp}, 0, X(L))^*_{\mathrm{laur}}$ onto the space $\mathrm{Laur}\,(V_{\mathbb{C}}, L, \mathcal{H})$. This isomorphism corresponds with the identity on $S_{\leftarrow}(V_L^{\perp}, X^0(L))$, via the isomorphisms of Lemma 10.4 and eq. (11.4).*

Proof: See [11], Appendix B, Lemma B.3. □

Remark 11.5 In the formulation of (c) we have used that the spaces $\mathcal{M}(V_{L\mathbb{C}}^{\perp}, 0, X(L))^*_{\mathrm{laur}}$ and $\mathcal{M}(V_{L\mathbb{C}}^{\perp}, 0, X^0(L))^*_{\mathrm{laur}}$ are equal, see Lemma 10.3.

We now assume that $\mathcal{H}$ is an X-configuration, and that $L \in \mathcal{A}$. If $a \in V_{L\mathbb{C}}^{\perp}$, then by $\mathcal{H}_L(a)$ we denote the collection of hyperplanes H' in L for which there exists a $H \in \mathcal{H}$ such that $H' = L \cap [(-a) + H]$. Thus, $\mathcal{H}_L(a) = (T_{-a}\mathcal{H})_L$ and we see that $\mathcal{H}_L(a)$ is an X_L-configuration. If $S \subset V_{L\mathbb{C}}^{\perp}$ is a finite subset, then

$$\mathcal{H}_L(S) = \cup_{a \in S} \mathcal{H}_L(a) \qquad (11.6)$$

is an X_L-configuration in L as well. The corresponding set of regular points in L equals

$$L \setminus \cup\mathcal{H}_L(S) = \{w \in L \mid \forall a \in S\, \forall H \in \mathcal{H}\colon\ a + w \in H \Rightarrow a + L \subset H\}.$$

Corollary 11.6 *Let $L \in \mathcal{A}$ and let $\mathcal{H}$ be an X-configuration. Let $S \subset V_{L\mathbb{C}}^{\perp}$ be a finite subset and let E be a complete locally convex space.*

(a) *For every $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ and each $w \in L \setminus \cup\mathcal{H}_L(S)$, there exists an open neighborhood Ω of S in $V_{L\mathbb{C}}^{\perp}$ such that the function $\varphi(w + \cdot\,)\colon z \mapsto \varphi(w+z)$ belongs to $\mathcal{M}(\Omega, X(L), E)$.*

(b) *Let $\mathcal{L} \in \mathcal{M}(V_{L\mathbb{C}}^{\perp}, X(L))^*_{\mathrm{laur}}$ be a Laurent functional supported at S. For every $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ the function $\mathcal{L}_*\varphi\colon L \setminus \cup\mathcal{H}_L(S) \to E$ defined by*

$$\mathcal{L}_*\varphi(w) := \mathcal{L}(\varphi(w + \cdot\,)) \tag{11.7}$$

belongs to $\mathcal{M}(L, \mathcal{H}_L(S), E)$. Finally, $\mathcal{L}_*$ is a continuous linear map from $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E)$ to $\mathcal{M}(L, \mathcal{H}_L(S), E)$. In fact, for every $d \in \mathbb{N}^{\mathcal{H}}$ there exists a $d' \in \mathbb{N}^{\mathcal{H}_L(S)}$, independent of E, such that $\mathcal{L}_*$ maps $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ continuously into $\mathcal{M}(L, \mathcal{H}_L(S), d', E)$.

Proof: It suffices to prove the result for S consisting of a single point a. Applying a translation by $-a$ if necessary, we may as well assume that $a = 0$. Then $\mathcal{H}_L(S) = \mathcal{H}_L(0) = \mathcal{H}_L$. Let $\mathcal{H}'$ be the union of $\mathcal{H}$ with $\mathcal{H}(L, X)$. Then $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, E) \subset \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}', E)$ and $(\mathcal{H}')_L = \mathcal{H}_L = \mathcal{H}_L(S)$, hence assertions (a) and (b) of Lemma 11.4 with $\mathcal{H}'$ in place of $\mathcal{H}$ imply assertion (a) and (b), except for the final statement about the continuity.

For the final statement of (b), we note that by Lemma 11.4(b), $\mathcal{L}_*$ is a Laurent operator $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}') \to \mathcal{M}(L, \mathcal{H}_L(S))$. Let $d\colon \mathcal{H} \to \mathbb{N}$ be a map. We extend d to $\mathcal{H}'$ by triviality on $\mathcal{H}' \setminus \mathcal{H}$. Then according to Lemma 11.3 there exists a map $d'\colon \mathcal{H}_L(S) \to \mathbb{N}$ such that for any complete locally convex space E the map

$$\mathcal{L}_*\colon \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}', d, E) \to \mathcal{M}(L, \mathcal{H}_L(S), d', E)$$

is continuous linear. Since d is zero on $\mathcal{H}' \setminus \mathcal{H}$, the first of these spaces equals $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H}, d, E)$ and the asserted continuity follows. $\square$

Lemma 11.7 *Let L, $\mathcal{H}$, S and $\mathcal{L}$ be as in Cor. 11.6, and fix $w \in L \setminus \cup\mathcal{H}_L(S)$. There exists a Laurent functional (in general not unique) $\mathcal{L}' \in \mathcal{M}(V_{\mathbb{C}}, X)^*_{\mathrm{laur}}$, supported in $w + S$, such that $\mathcal{L}'\varphi = \mathcal{L}(\varphi(w + \cdot\,))$ for all $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$.*

Proof: As in the proof of Cor. 11.6 we may assume that $S = \{0\}$. Let $\tilde{\mathcal{H}} = \mathcal{H} \cup \mathcal{H}(w, X)$. Then $\mathcal{L}_*\colon \varphi \mapsto \mathcal{L}(\varphi(w + \cdot\,))$ is a Laurent operator in $\mathrm{Laur}\,(V_{\mathbb{C}}, L, \tilde{\mathcal{H}})$, according to Lemma 11.4 (b). On the other hand, it follows from Lemma 10.5 (see Remark 10.6) that there exists a (in general not unique)

X_L-Laurent functional $\mathcal{L}''$ on L such that $\psi(w) = \mathcal{L}''(\psi_w)$ for each $\psi \in \mathcal{O}_w(L)$. The functional $\psi \mapsto \mathcal{L}''(\psi_w)$ is defined for $\psi \in \mathcal{M}(L, \tilde{\mathcal{H}}_L)$, and it may be viewed as a Laurent operator in Laur $(L, \{w\}, \tilde{\mathcal{H}}_L)$, which we denote by the same symbol $\mathcal{L}''$ (see [11] Appendix, Remark B.4). It now follows from [10], Lemma 1.8 that the composed map $\mathcal{L}'' \circ \mathcal{L}_*$ belongs to Laur $(V_\mathbb{C}, \{w\}, \tilde{\mathcal{H}})$ and hence by [11] Appendix, Remark B.4 it is given by an X-Laurent functional $\mathcal{L}'$, supported at w. In particular, for $\varphi \in \mathcal{M}(V_\mathbb{C}, \mathcal{H})$ we have from Lemma 11.4 (b) that $w \mapsto \mathcal{L}(\varphi(w + \,\cdot\,))$ is holomorphic in a neighborhood of w, hence its evaluation at w is obtained from the application of $\mathcal{L}''$ to it. Thus $\mathcal{L}(\varphi(w + \,\cdot\,)) = \mathcal{L}'\varphi$ for $\varphi \in \mathcal{M}(V_\mathbb{C}, \mathcal{H})$. $\square$

Recall from Section 10 that $\mathcal{M}(V_\mathbb{C}, X, E)$ is the union of the spaces $\mathcal{M}(V_\mathbb{C}, \mathcal{H}, E)$ with $\mathcal{H}$ an X-configuration.

Lemma 11.8 *Let $L \in \mathcal{A}$ and let $\mathcal{L} \in \mathcal{M}(V_{L\mathbb{C}}^\perp, X(L))^*_{\mathrm{laur}}$ be a Laurent functional. Then for any complete locally convex space E there exists a unique linear operator*

$$\mathcal{L}_*\colon \mathcal{M}(V_\mathbb{C}, X, E) \to \mathcal{M}(L, X_L, E)$$

that coincides on the subspace $\mathcal{M}(V_\mathbb{C}, \mathcal{H}, E)$ with the operator $\mathcal{L}_*$ defined in Corollary 11.6, for every X-configuration $\mathcal{H}$ in $V_\mathbb{C}$.

Proof: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two X-configurations. Let $S = \mathrm{supp}\,(\mathcal{L})$ and let, for $j = 1, 2$, the continuous linear operator $\mathcal{L}_*^j\colon \mathcal{M}(V_\mathbb{C}, \mathcal{H}_j, E) \to \mathcal{M}(L, \mathcal{H}_{jL}(S), E)$ be defined as in Corollary 11.6 with $\mathcal{H}_j$ in place of $\mathcal{H}$. Then it suffices to show that $\mathcal{L}_*^1$ and $\mathcal{L}_*^2$ coincide on the intersection of $\mathcal{M}(V_\mathbb{C}, \mathcal{H}_1, E)$ and $\mathcal{M}(V_\mathbb{C}, \mathcal{H}_2, E)$. That intersection equals $\mathcal{M}(V_\mathbb{C}, \mathcal{H}_1 \cap \mathcal{H}_2, E)$. Let φ be a function in the latter space, then from the defining formula (11.7) it follows that $\mathcal{L}_*^1\varphi = \mathcal{L}_*^2\varphi$ on the intersection of the sets $L \setminus \cup\mathcal{H}_{jL}(S)$, for $j = 1, 2$. This implies that $\mathcal{L}_*^1\varphi$ and $\mathcal{L}_*^2\varphi$ coincide as elements of $\mathcal{M}(L)$. $\square$

We end this section with another useful consequence.

Lemma 11.9 *Let $\mathcal{L} \in \mathcal{M}(V_{L\mathbb{C}}^\perp, X(L))^*_{\mathrm{laur}}$. Let the finite subset $\widetilde{X}$ of $V \times V \setminus \{(0,0)\}$ be defined by $\widetilde{X} = (X \times \{0\}) \cup (\{0\} \times X)$. If $\Phi \in \mathcal{M}(V_\mathbb{C} \times V_\mathbb{C}, \widetilde{X})$, then*

$$\Psi\colon (w_1, w_2) \mapsto \mathcal{L}(\Phi(\,\cdot\, + w_1, \,\cdot\, + w_2))$$

defines a function in $\mathcal{M}(L \times L, \widetilde{X}_L)$, where $\widetilde{X}_L = (X_L \times \{c(L)\}) \cup (\{c(L)\} \times X_L)$. In particular, the pull-back of Ψ under the diagonal embedding $j\colon L \to L \times L$ belongs to the space $\mathcal{M}(L, X_L)$.

Proof: Equip $V_L^\perp \times V_L^\perp$ with one half times the direct sum inner product. Then the diagonal embedding $\iota\colon z \mapsto (z, z)$ is an isometry of $V_L^\perp$ into $V_L^\perp \times V_L^\perp$. Its adjoint is the map $p\colon (z_1, z_2) \mapsto \frac{1}{2}(z_1 + z_2)$ from $V_L^\perp \times V_L^\perp$ onto $V_L^\perp$. The intersection $\widetilde{X}(L) := \widetilde{X} \cap (V_L^\perp \times V_L^\perp)$ equals $(X(L) \times \{0\}) \cup (\{0\} \times X(L))$. Its image under p is given by $\widetilde{X}(L)_0 = \frac{1}{2}X(L)$. Thus, according to Lemma

10.10, the space of $\widetilde{X}(L)_0$-Laurent functionals on $V_{L\mathbb{C}}^\perp$ is equal to the space of $X(L)$-Laurent functionals on $V_{L\mathbb{C}}^\perp$. Hence, according to Lemma 10.14 and the remark following its proof, we have an associated push-forward map ι_* from $\mathcal{M}(V_{L\mathbb{C}}^\perp, X(L))_{\mathrm{laur}}^*$ to $\mathcal{M}(V_{L\mathbb{C}}^\perp \times V_{L\mathbb{C}}^\perp, \widetilde{X}(L))_{\mathrm{laur}}^*$.

For generic $w_1, w_2 \in L$ we define the meromorphic function $\Phi^{(w_1,w_2)}$ on $V_{L\mathbb{C}}^\perp \times V_{L\mathbb{C}}^\perp$ by $\Phi^{(w_1,w_2)}(z_1,z_2) = \Phi(w_1+z_1, w_2+z_2)$. The definition of Ψ may now be rewritten as $\Psi(w_1,w_2) = \mathcal{L}[\iota^*(\Phi^{(w_1,w_2)})]$. By (10.6) it follows that $\Psi(w_1,w_2) = \iota_*(\mathcal{L})(\Phi^{(w_1,w_2)})$, or, equivalently, in the notation of Lemma 11.8,

$$\Psi = [\iota_*(\mathcal{L})]_*\Phi.$$

We now observe that $\widetilde{X}_L = (\widetilde{X})_{L\times L}$. Hence it follows by application of Lemma 11.8. that $\Psi \in \mathcal{M}(L\times L, \tilde{X}_L)$. There exists an $\tilde{X}_L$-configuration $\tilde{\mathcal{H}}$ in $L\times L$ such that $\Psi \in \mathcal{M}(L \times L, \tilde{\mathcal{H}})$. Any hyperplane $\tilde{H} \in \tilde{\mathcal{H}}$ is of the form $\tilde{H} = H \times L$ or $\tilde{H} = L \times H$, with H an X_L-hyperplane in L. In both cases $j^{-1}(\tilde{H}) = H$. It now follows that $j^{-1}(\tilde{\mathcal{H}})$ is an X_L-configuration in L, and that $j^*\Psi \in \mathcal{M}(L, X_L)$. $\square$

12 Analytic families of a special type

In this section we introduce a space $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \delta)$ of analytic families of $\mathbb{D}(\mathrm{X})$-finite τ-spherical functions whose singular locus is a Σ-configuration. The definition of this space is motivated by the fact that it contains the families obtained from applying Laurent functionals to Eisenstein integrals related to a minimal σ-parabolic subgroup, as we shall see in the following sections, and by the fact that the vanishing theorem is applicable, see Theorem 12.10.

In this section we fix a choice Σ^+ of positive roots for Σ and denote by P_0 the associated minimal standard σ-parabolic subgroup.

Definition 12.1 *Let $Q \in \mathcal{P}_\sigma$ and let $Y \subset {}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ be a finite subset. We define*

$$C_{Q,Y}^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau) \tag{12.1}$$

to be the space of functions $f\colon \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \times \mathrm{X}_+ \to V_\tau$, meromorphic in the first variable, for which there exist a constant $k \in \mathbb{N}$, a $\Sigma_r(Q)$-hyperplane configuration $\mathcal{H}$ in $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and a function $d\colon \mathcal{H} \to \mathbb{N}$ such that the following conditions are fulfilled.

(a) *The function $\lambda \mapsto f_\lambda$ belongs to $\mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(\mathrm{X}_+ : \tau))$.*

(b) *For every $P \in \mathcal{P}_\sigma^{\min}$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$ there exist functions $q_{s,\xi}(P, v \,|\, f)$ in $P_k(\mathfrak{a}_\mathrm{q}) \otimes \mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(\mathrm{X}_{0,v} : \tau_\mathrm{M}))$, for $s \in W/W_Q$ and $\xi \in -sW_QY + \mathbb{N}\Delta(P)$, with the following property. For all $\lambda \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}$, $m \in \mathrm{X}_{0,v}$ and $a \in A_\mathrm{q}^+(P)$,*

$$f_\lambda(mav) = \sum_{s\in W/W_Q} a^{s\lambda-\rho_P} \sum_{\xi\in -sW_QY+\mathbb{N}\Delta(P)} a^{-\xi}\, q_{s,\xi}(P, v \,|\, f, \log a)(\lambda, m), \tag{12.2}$$

where the $\Delta(P)$-exponential polynomial series of each inner sum converges neatly on $A_{\mathrm{q}}^{+}(P)$.

(c) *For every $P \in \mathcal{P}_{\sigma}^{\min}$, $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and $s \in W/W_Q$, the series*

$$\sum_{\xi \in -sW_Q Y + \mathbb{N}\Delta(P)} a^{-\xi} q_{s,\xi}(P, v \,|\, f, \log a)$$

converges neatly on $A_{\mathrm{q}}^{+}(P)$, as an exponential polynomial series with coefficients in the space $\mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*}, \mathcal{H}, d, C^{\infty}(\mathrm{X}_{0,v} : \tau_{\mathrm{M}}))$.

Finally, we define

$$C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau) := C_{P_0,\{0\}}^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau). \tag{12.3}$$

Remark 12.2 Note the analogy between the above definition and Definition 7.1. In fact, let $\Omega = \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*} \setminus \cup\mathcal{H}$, then it follows immediately from the definitions that the restriction of f to $\Omega \times \mathrm{X}_+$ belongs to $C_{Q,Y}^{\mathrm{ep}}(\mathrm{X}_+ : \tau : \Omega)$. Moreover, it follows from Lemma 7.3 that the functions $q_{s,\mu}(P, v \,|\, f)$ introduced above are unique, and that the notation used here is consistent with the notation in Definition 7.1. The precise relation between the definitions is given in Lemma 12.5 below.

Remark 12.3 In analogy with Remark 7.2 we note that the space (12.1) depends on Q through its σ-split component $A_{Q\mathrm{q}}$. Moreover, it suffices in the above definition to require conditions (b) and (c) for a fixed $P \in \mathcal{P}_{\sigma}^{\min}$ and all v in a given set $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ of representatives for $W/W_{K\cap H}$. Alternatively, it suffices to require those conditions for a fixed given $v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and each $P \in \mathcal{P}_{\sigma}^{\min}$.

Finally, we note that $\mathfrak{a}_{P_0\mathrm{q}} = \mathfrak{a}_{\mathrm{q}}$, hence ${}^*\mathfrak{a}_{P_0} = \{0\}$. Thus, if $Q = P_0$, we only need to consider the finite set $Y = \{0\}$. This explains the limitation in (12.3).

It follows from Remark 12.2 that the following definition of the notion of asymptotic degree is in accordance with the definition of the similar notion in Definition 7.1.

Definition 12.4 *Let $f \in C_{Q,Y}^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$. We define the asymptotic degree of f, denoted $\deg_{\mathrm{a}}(f)$, to be the smallest integer k for which there exist $\mathcal{H}, d$ such that the conditions of Definition 12.1 are fulfilled. Moreover, we denote by $\mathcal{H}_f$ the smallest $\Sigma_r(Q)$-configuration in $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*}$ such that the conditions of Definition 12.1 are fulfilled with $k = \deg_{\mathrm{a}}(f)$ and for some $d : \mathcal{H}_f \to \mathbb{N}$. These choices being fixed, we denote by d_f the $\preceq$-minimal map $\mathcal{H}_f \to \mathbb{N}$ for which the conditions of the definition are fulfilled. Finally, we put $\mathrm{reg}_{\mathrm{a}}(f) := \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^{*} \setminus \cup\mathcal{H}_f$.*

If $Q \in \mathcal{P}_{\sigma}$, we denote by $\Sigma_{r0}(Q)$ the set of indivisible roots in $\Sigma_r(Q)$, i.e., the roots $\alpha \in \Sigma_r(Q)$ with $]0,1]\alpha \cap \Sigma_r(Q) = \{\alpha\}$. Moreover, we put $\Sigma_0^{+} = \Sigma_{r0}(P_0)$.

Let $\mathcal{H}$ be a $\Sigma_r(Q)$-configuration in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $d\colon \mathcal{H} \to \mathbb{N}$ a map. If $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a bounded subset, we define $\pi_{\omega,d}$ as in (11.1) with $V = \mathfrak{a}^*_{Q\mathrm{q}}$, $X = \Sigma_r(Q)$ and $X^0 = \Sigma_{r0}(Q)$.

Lemma 12.5 *Let $Q \in \mathcal{P}_\sigma$, $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset, $\mathcal{H}$ a $\Sigma_r(Q)$-configuration in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $d \in \mathbb{N}^{\mathcal{H}}$. Assume that $f \in \mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, C^\infty(\mathrm{X}_+ : \tau))$. Then the following two conditions are equivalent.*

(a) *The function f belongs to $C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$ and satisfies $\mathcal{H}_f \subset \mathcal{H}$ and $d_f \preceq d$.*

(b) *For every non-empty bounded open subset $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the function $f_{\pi_{\omega,d}}\colon (\lambda, x) \mapsto \pi_{\omega,d}(\lambda) f(\lambda, x)$, $\omega \times \mathrm{X}_+ \to V_\tau$ belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \omega)$.*

Moreover, if one of the above equivalent conditions is fulfilled, then for every non-empty bounded open subset $\omega \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and all $P \in \mathcal{P}^{\min}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$, $s \in W/W_Q$ and $\xi \in -sW_Q Y + \mathbb{N}\Delta(P)$,

$$q_{s,\xi}(P, v \,|\, f_{\pi_{\omega,d}}) = \pi_{\omega,d}\, q_{s,\xi}(P, v \,|\, f), \tag{12.4}$$

where on the right-hand side we have identified $\pi_{\omega,d}$ with the function $1 \otimes \pi_{\omega,d} \otimes 1$ in $P(\mathfrak{a}_\mathrm{q}) \otimes \mathcal{O}(\omega) \otimes C^\infty(\mathrm{X}_{0,v} : \tau)$.

Proof: Assume that (a) holds and that $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a non-empty bounded open subset. Put $\pi = \pi_{\omega,d}$ and $f_\pi = f_{\pi_{\omega,d}}$. It follows from Definition 12.1 (a) that $f_\pi\colon \omega \times \mathrm{X}_+ \to V_\tau$ is smooth and that $f_{\pi\lambda}$ is τ-spherical for every $\lambda \in \omega$. Thus, it remains to verify conditions (b) and (c) of Definition 7.1 for f_π. Let $P \in \mathcal{P}^{\min}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$. For $s \in W/W_Q$ and $\xi \in -sW_Q Y + \mathbb{N}\Delta(P)$ we define

$$q'_{s,\xi}(P, v \,|\, f_\pi, X, \lambda, m) := \pi(\lambda) q_{s,\xi}(P, v \,|\, f, X, \lambda, m).$$

Then conditions (b) and (c) of Definition 7.1, with $k = \deg_\mathrm{a} f$ and with $q'_{s,\xi}$ in place of $q_{s,\xi}$, follow from the similar conditions of Definition 12.1. Thus, it follows that $f_\pi \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \omega)$ and that (12.4) holds for all $P \in \mathcal{P}^{\min}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$, $s \in W$ and $\xi \in -sW_Q Y + \mathbb{N}\Delta(P)$.

Now assume that (b) holds, then it suffices to show that (a) holds. Let ω be a bounded non-empty open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Then it follows from Definition 7.1 that the function $f_\pi = f_{\pi_{\omega,d}}\colon \omega \times \mathrm{X}_+ \to V_\tau$ is smooth; moreover, from condition (a) of the mentioned definition it follows that $f_{\pi,\lambda}$ is τ-spherical for every $\lambda \in \omega$. Hence the map $\lambda \mapsto f_\pi$ belongs to $\mathcal{O}(\omega, C^\infty(\mathrm{X}_+ : \tau))$. Since ω was arbitrary, this implies that $\lambda \mapsto f_\lambda$ belongs to $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}, d, C^\infty(\mathrm{X}_+ : \tau))$. Hence f satisfies condition (a) of Definition 12.1. Let now $P \in \mathcal{P}^{\min}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$. Then it remains to establish conditions (b) and (c) of that definition.

If ω is a non-empty bounded open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then obviously the restriction to $\omega \setminus \cup\mathcal{H}$ of the function f_π belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \omega \setminus \cup\mathcal{H})$. Moreover, since $\pi_{\omega,d}$ is nowhere zero on $\omega \setminus \cup\mathcal{H}$, it follows from division by $\pi_{\omega,d}$ that the restriction $f|_{(\omega\setminus\cup\mathcal{H})\times \mathrm{X}_+}$ belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \omega \setminus \cup\mathcal{H})$. Hence, in view of Lemma 7.5, the function f belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$, where $\Omega := \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}$. Let $k = \deg_\mathrm{a} f$.

It follows from the division by $\pi_{\omega,d}$, that for every $s \in W$ and $\xi \in -sW_Q Y + \mathbb{N}\Delta(P)$,

$$\pi_{\omega,d}(\lambda) q_{s,\xi}(P, v \,|\, f, \,\cdot\,, \lambda) = q_{s,\xi}(P, v \,|\, f_\pi, \,\cdot\,, \lambda), \qquad (\lambda \in \omega \setminus \cup\mathcal{H}).$$

In particular, the function $(X, \lambda) \mapsto \pi_{\omega,d}(\lambda) q_{s,\xi}(P, v \,|\, f, X, \lambda)$ belongs to the space $P_k(\mathfrak{a}_{\mathrm{q}}) \otimes \mathcal{O}(\omega, C^\infty(\mathrm{X}_{0,v} : \tau_{\mathrm{M}}))$. Since ω is arbitrary, this implies that f satisfies condition (b) of Definition 12.1.

From condition (c) of Definition 7.1 with f_π and ω in place of f and Ω, respectively, it follows that, for $s \in W$, the series

$$\sum_{\xi \in -sW_Q Y + \mathbb{N}\Delta(P)} a^{-\xi} \pi_{\omega,d}(\lambda)\, q_{s,\xi}(P, v \,|\, f, \log a, \lambda)$$

converges neatly on $A_{\mathrm{q}}^+(P)$ as a $\Delta(P)$-exponential polynomial series with coefficients in $\mathcal{O}(\omega, C^\infty(\mathrm{X}_{0,v} : \tau))$. Since ω was arbitrary, it follows from the definition of the topology on $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}, d, C^\infty(\mathrm{X}_{0,v} : \tau_{\mathrm{M}}))$ (see Section 11) that f satisfies condition (c) of Definition 12.1. $\square$

Lemma 12.6 *Let $f \in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$ and $D \in \mathbb{D}(\mathrm{X})$. Then $Df \in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$. Moreover, $\mathcal{H}_{Df} \subset \mathcal{H}_f$, $d_{Df} \preceq d_f$ and $\deg_{\mathrm{a}} Df \leq \deg_{\mathrm{a}} f$.*

Proof: This follows from a straightforward combination of Lemma 12.5 with Proposition 7.6. $\square$

If $f \in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$, then by Remark 12.2 the function f belongs to $C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$, with $\Omega = \mathrm{reg}_{\mathrm{a}} f$. Let $k = \deg_{\mathrm{a}} f$. For $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_{\mathrm{q}})$, $\sigma \in W/\!\sim_{P|Q}$ and $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$, let $q_{\sigma,\xi}(P, v \,|\, f) \in P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{O}(\Omega, C^\infty(\mathrm{X}_{P,v,+} : \tau_P))$ be the function defined in Theorem 7.7.

Lemma 12.7 *Let $Q \in \mathcal{P}_\sigma$ and $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset. Assume that $f \in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$ and put $k = \deg_{\mathrm{a}} f$. Let $P \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$. Then, for every $\lambda \in \mathrm{reg}_{\mathrm{a}} f$, the set $\mathrm{Exp}(P, v \,|\, f_\lambda)$ is contained in $W(\lambda + Y)|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \mathbb{N}\Delta_r(P)$. Moreover, let $\sigma \in W/\!\sim_{P|Q}$. Then*

(a) *for every $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$,*

$$q_{\sigma,\xi}(P, v \,|\, f) \in P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}_f, d_f, C^\infty(\mathrm{X}_{P,v,+} : \tau_P));$$

(b) *for every $R > 1$, the series*

$$\sum_{\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)} a^{-\xi} q_{\sigma,\xi}(P, v \,|\, f, \log a)$$

converges neatly on $A^+_{P\mathrm{q}}(R^{-1})$ as a $\Delta_r(P)$-exponential polynomial series with coefficients in $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}_f, d_f, C^\infty(\mathrm{X}_{P,v,+}[R] : \tau_P))$.

Proof: Let $\Omega = \mathrm{reg}_{\mathrm{a}} f$. Then $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$. It follows from Theorem 7.7 that the assertion about the (P, v)-exponents of f_λ holds. That (a) and (b) hold can be seen as in the last part of the proof of Lemma 12.5, with the reference to Definition 7.1 replaced by reference to Theorem 7.7. $\square$

The following definition is the analogue for $C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$ of Definitions 9.1 and 9.5.

Definition 12.8 *Let $Q \in \mathcal{P}_\sigma$ and $\delta \in \mathrm{D}_Q$. Then for $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ a finite subset we define*

$$\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)$$

to be the space of functions $f \in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$ (see Definition 12.1) such that, for all $\lambda \in \mathrm{reg}_{\mathrm{a}}(f)$, the function $f_\lambda \colon x \mapsto f(\lambda, x)$ is annihilated by the cofinite ideal $I_{\delta,\lambda}$. Moreover, we define

$$\mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta) := \bigcup_{Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \text{ finite}} \mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta).$$

The spaces

$$\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}, \quad \mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}$$

are defined to be the spaces of functions f in $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)$, resp. $\mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)$, for which the condition in Definition 9.5 is satisfied by the restriction to $\Omega = \mathrm{reg}_{\mathrm{a}} f$.

Finally, we define

$$\mathcal{E}^{\mathrm{hyp}}_{0}(\mathrm{X}_+ : \tau : \delta) := \mathcal{E}^{\mathrm{hyp}}_{P_0}(\mathrm{X}_+ : \tau : \delta), \quad \mathcal{E}^{\mathrm{hyp}}_{0}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}} := \mathcal{E}^{\mathrm{hyp}}_{P_0}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}$$

for $\delta \in D_{P_0}$.

Remark 12.9 Combining Lemmas 12.5 and 9.4 we see that, in the above definition of $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)$, it suffices to require that $I_{\delta,\lambda}$ annihilates f_λ for λ in a non-empty open subset of $\mathrm{reg}_{\mathrm{a}}(f)$.

We now come to a special case of the vanishing theorem that will be particularly useful in the following. Let ${}^Q\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ be a complete set of representatives for $W_Q \backslash W / W_{K\cap H}$.

Theorem 12.10 (A special case of the vanishing theorem) *Let $Q \in \mathcal{P}_\sigma$ and let $\delta \in D_Q$. Let $f \in \mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}$ and let Ω' be a non-empty open subset of $\mathrm{reg}_{\mathrm{a}} f$. If*

$$\lambda - \rho_Q \notin \mathrm{Exp}(Q, u \mid f_\lambda)$$

for each $u \in {}^Q\mathcal{W}$ and all $\lambda \in \Omega'$, then $f = 0$.

Proof: Put $\Omega = \mathrm{reg}_{\mathrm{a}}(f)$. It follows immediately from the definitions that the restriction f_Ω of f to Ω is a family in $\mathcal{E}_Q(\mathrm{X}_+ : \tau : \Omega : \delta)_{\mathrm{glob}}$. Moreover, being the complement of a locally finite collection of hyperplanes, Ω is Q-distinguished in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. It follows that f_Ω satisfies all hypothesis of Theorem 9.10; hence $f_\Omega = 0$ and hence $f = 0$. $\square$

13 Action of Laurent functionals on analytic families

Let $Q \in \mathcal{P}_\sigma$ be fixed. We shall discuss the application of a Laurent functional $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$, to families $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$. More precisely, we want to set up natural conditions on f under which the family obtained from applying $\mathcal{L}$ to f belongs to $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}$, so that Theorem 12.10 is applicable.

Given a Σ-configuration $\mathcal{H}$ in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and a finite subset $S \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ we define the $\Sigma_r(Q)$-configuration $\mathcal{H}_Q(S) = \mathcal{H}_{\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}}(S)$ as in (11.6), with $V = \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, $X = \Sigma$, and $L = \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Thus, for $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ we have

$$\nu \notin \cup\mathcal{H}_Q(S) \iff [\, \forall \lambda \in S\, \forall H \in \mathcal{H}\colon \quad \lambda + \nu \in H \Rightarrow \lambda + \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \subset H \,].$$

We recall from Lemma 11.8 that a Laurent functional $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$ induces a linear operator

$$\mathcal{L}_*\colon \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma, U) \to \mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_r(Q), U), \tag{13.1}$$

for any complete locally convex space U.

Lemma 13.1 *Let $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$ and put $Y = \mathrm{supp}\,\mathcal{L}$. Let $\mathcal{H}$ be a Σ-configuration in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, and let $\mathcal{H}' = \mathcal{H}_Q(Y)$. Then for every map $d\colon \mathcal{H} \to \mathbb{N}$ there exists a map $d'\colon \mathcal{H}' \to \mathbb{N}$ such that, for every complete locally convex space U, the linear map (13.1) restricts to a continuous linear operator*

$$\mathcal{L}_*\colon \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d, U) \to \mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}', d', U),$$

Proof: This follows immediately from Corollary 11.6. $\square$

For the formulation of the next result it will be convenient to introduce a particular linear map. Let $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$ and let $\lambda_0 \in Y := \mathrm{supp}\,\mathcal{L}$. Let $\mathcal{L}_{\lambda_0} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$ be the Laurent functional supported at λ_0, defined as in Remark 10.9, and let U be a complete locally convex space. If $P \in \mathcal{P}_\sigma$ and $s \in W_P\backslash W$, then we define the linear operator $\mathcal{L}^{P,s}_{\lambda_0 *}$ from $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma, U)$ into $C(\mathfrak{a}_{P\mathrm{q}}, \mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_r(Q), U))$ by the formula

$$\mathcal{L}^{P,s}_{\lambda_0 *}\varphi(X, \nu) = e^{-s(\lambda_0 + \nu)(X)} \mathcal{L}_{\lambda_0 *}[e^{s(\,\cdot\,)(X)} \varphi(\,\cdot\,)](\nu), \tag{13.2}$$

for $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, U)$, $X \in \mathfrak{a}_{P\mathrm{q}}$ and $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}_Q(Y)$.

If $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$, then f, viewed as the function $\lambda \mapsto f_\lambda$, belongs to the complete locally convex space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}_f, d_f, C^\infty(\mathrm{X}_+ : \tau))$. Accordingly,

$$\mathcal{L}_* f \in \mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}', d', C^\infty(\mathrm{X}_+ : \tau)), \tag{13.3}$$

where $\mathcal{H}' = \mathcal{H}_{fQ}(Y)$ and $d' \colon \mathcal{H}' \to \mathbb{N}$ is associated with $\mathcal{L}, \mathcal{H}_f$ and d_f as in Lemma 13.1. We note that by definition

$$\mathcal{L}_* f(\nu, x) = \mathcal{L}[f(\,\cdot\, + \nu, x)], \qquad (\nu \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}',\ x \in \mathrm{X}_+). \tag{13.4}$$

Proposition 13.2 *Let $Q \in \mathcal{P}_\sigma$ and let $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}_{Q\mathrm{q}}, \Sigma_Q)_{\mathrm{laur}}^*$ be a Laurent functional with support contained in the finite subset $Y \subset {}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$. Assume that $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$, and let $k = \deg_{\mathrm{a}} f$.*

(a) *The function $\mathcal{L}_* f$, defined as in (13.4), belongs to the space $C_{Q,Y}^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$. Moreover, $\mathcal{H}_{\mathcal{L}_* f} \subset \mathcal{H}' = \mathcal{H}_{fQ}(Y)$ and $\deg_{\mathrm{a}} \mathcal{L}_* f \leq k + k'$, with $k' \in \mathbb{N}$ a constant only depending on $\mathcal{L}, \mathcal{H}_f$ and d_f.*

(b) *Let $P \in \mathcal{P}_\sigma, v \in N_K(\mathfrak{a}_{\mathrm{q}})$. Then, for $\sigma \in W/{\sim_{P|Q}}$ and $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$,*

$$q_{\sigma,\xi}(P, v \mid \mathcal{L}_* f, X, \nu) = \sum_{\lambda \in Y} \sum_{\substack{s \in W_P \backslash W,\, [s] = \sigma \\ s\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + \xi \in \mathbb{N}\Delta_r(P)}} \mathcal{L}_{\lambda *}^{P,s} \left[q_{s, s\lambda|_{\mathfrak{a}_{P\mathrm{q}}} + \xi}(P, v \mid f)(X, \,\cdot\,) \right](\nu, X), \tag{13.5}$$

for all $X \in \mathfrak{a}_{P\mathrm{q}}$ and $\nu \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}'$. In particular,

$$\begin{aligned} &\mathrm{Exp}(P, v \mid (\mathcal{L}_* f)_\nu) \\ &\quad \subset \{ s(\nu + \lambda)|_{\mathfrak{a}_{P\mathrm{q}}} - \rho_P - \mu \mid s \in W, \lambda \in Y, \mu \in \mathbb{N}\Delta_r(P), q_{s,\mu}(P, v \mid f) \neq 0 \}. \end{aligned}$$

Remark 13.3 Note that the index set of the inner sum in (13.5) may be empty. We agree that such a sum should be interpreted as zero.

The following lemma prepares for the proof of the proposition.

Lemma 13.4 *Let $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \Sigma_Q)_{\mathrm{laur}}^*$ be a Laurent functional with support contained in the finite set $Y \subset {}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$. Let $\mathcal{H}$ be a Σ-configuration in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $d \colon \mathcal{H} \to \mathbb{N}$ a map. Let $\mathcal{H}' = \mathcal{H}_Q(Y)$ and $d' \colon \mathcal{H}' \to \mathbb{N}$ be as in Lemma 13.1. There exists a natural number $k' \in \mathbb{N}$ with the following property.*

For every $\lambda_0 \in Y$, every $P \in \mathcal{P}_\sigma$, each $s \in W_P \backslash W$ and any complete locally convex space U, the operator $\mathcal{L}_{\lambda_0 *}^{P,s}$ restricts to a continuous linear map

$$\mathcal{L}_{\lambda_0 *}^{P,s} \colon\ \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, U) \to P_{k'}(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}', d', U).$$

Proof: For a fixed $X \in \mathfrak{a}_{P\mathrm{q}}$, multiplication by the holomorphic function $e^{s(\,\cdot\,)(X)} \colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$ yields a continuous linear endomorphism of the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, U)$;

similarly, multiplication by the holomorphic function $e^{-s(\lambda_0+\,\cdot\,)(X)}: \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \to \mathbb{C}$ yields a continuous linear endomorphism of $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}', d', U)$. It now follows from (13.2) that for a fixed $X \in \mathfrak{a}_{P\mathrm{q}}$, the function $\mathcal{L}^{P,s}_{\lambda_0 *}\varphi(X)$ belongs to the space $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}', d', U)$ and depends continuously on φ. Thus, it remains to establish the polynomial dependence on X.

For any Σ-hyperplane $H \subset \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ we denote by α_H the root from Σ_0^+ such that H is a translate of $\alpha_{H\mathbb{C}}^\perp$. Let $\Sigma^+_{Q,0} := \Sigma_Q \cap \Sigma_0^+$ and let $d_0: \Sigma^+_{Q,0} \to \mathbb{N}$ be defined by $d_0(\alpha) = d(\alpha^\perp + \lambda_0)$; thus $d_0(\alpha) = 0$ if $\alpha^\perp + \lambda_0 \notin \mathcal{H}$. We define $\pi_0 = \pi_{\lambda_0, d_0}$ as in (10.1) with ${}^*\mathfrak{a}^*_{Q\mathrm{q}}, \lambda_0, \Sigma^+_{Q,0}$ and d_0 in place of V, a, X and d, respectively. If $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d, U)$, then for $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}'$, the germ of the function $\varphi^\nu: \lambda \mapsto \varphi(\lambda + \nu)$ at λ_0 belongs to $\pi_0^{-1}\mathcal{O}_{\lambda_0}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, U)$. Hence there exists a constant coefficient differential operator $u_0 \in S({}^*\mathfrak{a}^*_{Q\mathrm{q}})$, independent of U, such that

$$\mathcal{L}_{\lambda_0 *}\varphi(\nu) = u_0[\pi_0(\,\cdot\,)\varphi(\,\cdot\, + \nu)](\lambda_0), \qquad (\nu \in \mathfrak{a}^*_{Q\mathrm{q}} \setminus \cup\mathcal{H}'), \tag{13.6}$$

for any $\varphi \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d, U)$. Inserting (13.6) in (13.2) we find that

$$\begin{aligned} \mathcal{L}^{P,s}_{\lambda_0 *}\varphi(X, \nu) &= e^{-s(\lambda_0+\nu)(X)} u_0[e^{s(\,\cdot\,+\nu)(X)}\pi_0(\,\cdot\,)\varphi(\,\cdot\, + \nu)](\lambda_0) \\ &= e^{-s(\lambda_0)(X)} u_0[e^{s(\,\cdot\,)(X)}\pi_0(\,\cdot\,)\varphi(\,\cdot\, + \nu)](\lambda_0). \end{aligned}$$

By application of the Leibniz rule it finally follows that this expression is polynomial in the variable X of degree at most $k' := \mathrm{order}(u_0)$. □

Proof of Proposition 13.2: By linearity we may assume that $\mathrm{supp}\,\mathcal{L}$ consists of a single point $\lambda_0 \in {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Let $\mathcal{H} = \mathcal{H}_f$ and $d = d_f$, and let $d': \mathcal{H}' \to \mathbb{N}$ and $k' \in \mathbb{N}$ be associated as in Lemmas 13.1 and 13.4. We will establish parts (a), (b) and (c) of Definition 12.1 for $\mathcal{L}_* f$ with k, $\mathcal{H}$ and d replaced by $k + k'$, $\mathcal{H}'$ and d'. Note that part (a) was observed already in (13.3). Put $\Omega := \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}'$. Then, in particular, the function $\mathcal{L}_* f: \Omega \times \mathrm{X}_+ \to V_\tau$ is smooth.

We will establish parts (b) and (c) of Definition 12.1 by obtaining an exponential polynomial expansion for $(\mathcal{L}_* f)_\nu$, for $\nu \in \Omega$, along $P \in \mathcal{P}_\sigma^{\min}$. However, having the proof of (13.5) in mind, we assume only $P \in \mathcal{P}_\sigma$ at present. Let $v \in N_K(\mathfrak{a}_\mathrm{q})$. Then $f \in C^{\mathrm{ep}}_{P_0,\{0\}}(\mathrm{X}_+ : \tau : \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H})$ by Remark 12.2. Hence by Lemma 12.7 and (7.13) we obtain, for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}$,

$$f(\lambda, mav) = \sum_{s \in W_P \backslash W} f_s(\lambda, a, m), \qquad (m \in \mathrm{X}_{P,v,+},\ a \in A^+_{P\mathrm{q}}(R_{P,v}(m)^{-1})), \tag{13.7}$$

where the functions f_s on the right-hand side are defined by

$$f_s(\lambda, a, m) = a^{s\lambda - \rho_P} \sum_{\mu \in \mathbb{N}\Delta_r(P)} a^{-\mu} q_{s,\mu}(P, v \,|\, f)(\log a, \lambda, m). \tag{13.8}$$

Here the functions $q_{s,\mu}(P, v \,|\, f)$ belong to the space $P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d, C^\infty(\mathrm{X}_{P,v,+} : \tau_P))$. By Lemma 12.7 (b), for every $R > 1$ the series in (13.8) converges neatly on $A^+_{P\mathrm{q}}(R^{-1})$ as a series with coefficients in $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H}, d, C^\infty(\mathrm{X}_{P,v,+}[R] : \tau_P))$. By (13.2) we have, for $\nu \in \Omega$, $m \in \mathrm{X}_{P,v,+}[R]$ and $a \in A^+_{P\mathrm{q}}(R^{-1})$

$$\mathcal{L}_*(f_s)(\nu, a, m) = a^{s(\lambda_0+\nu)-\rho_P} \mathcal{L}^{P,s}_{\lambda_0 *}[\sum_{\mu \in \mathbb{N}\Delta_r(P)} a^{-\mu} q_{s,\mu}(P, v \,|\, f)(\log a, \,\cdot\,, m)](\log a, \nu).$$

It follows from Lemma 13.4 that $\mathcal{L}^{P,s}_{\lambda_0 *}$ may be applied term by term to the series. Moreover, the resulting series is neatly convergent on $A^+_{P\mathrm{q}}(R^{-1})$ as a $\Delta_r(P)$-exponential polynomial series with coefficients in $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}', d', C^\infty(\mathrm{X}_{P,v,+}[R] : \tau_P))$.

The application of $\mathcal{L}_*$ thus leads to the following identity,

$$\mathcal{L}_*(f_s)(\nu, a, m) = a^{s(\lambda_0+\nu)-\rho_P} \sum_{\mu \in \mathbb{N}\Delta_r(P)} a^{-\mu} q^{\mathcal{L}}_{s,\mu}(P, v \,|\, f)(\log a, \nu, m), \tag{13.9}$$

where the function $q^{\mathcal{L}}_{s,\mu}(P, v \,|\, f) \colon \mathfrak{a}_{P\mathrm{q}} \times \Omega \to C^\infty(\mathrm{X}_{P,v,+} : \tau_P)$ is given by

$$q^{\mathcal{L}}_{s,\mu}(P, v \,|\, f)(\log a, \nu) = \mathcal{L}^{P,s}_{\lambda_0 *}[q_{s,\mu}(P, v \,|\, f, \log a, \,\cdot\,)](\log a, \nu). \tag{13.10}$$

Using Lemma 13.4 we deduce that

$$q^{\mathcal{L}}_{s,\mu}(P, v \,|\, f) \in P_{k+k'}(\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \mathcal{H}', d', C^\infty(\mathrm{X}_{P,v,+} : \tau_P)).$$

Combining (13.9) with (13.7) we obtain an exponential polynomial expansion along (P, v) for the τ-spherical function $(\mathcal{L}_* f)_\nu$ as

$$(\mathcal{L}_* f)_\nu(mav) = \sum_{s \in W_P \backslash W} a^{s(\lambda_0+\nu)-\rho_P} \sum_{\mu \in \mathbb{N}\Delta_r(P)} a^{-\mu} q^{\mathcal{L}}_{s,\mu}(P, v \,|\, f)(\log a, \nu, m). \tag{13.11}$$

If $s \in W_P \backslash W$ and $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then $s\nu|_{\mathfrak{a}_{P\mathrm{q}}} = [s]\nu|_{\mathfrak{a}_{P\mathrm{q}}}$, where $[s]$ denotes the class of s in $W/\!\sim_{P|Q}$. It follows that the series in (13.11) may be rewritten as

$$\sum_{\sigma \in W/\sim_{P|Q}} a^{\sigma\nu - \rho_P} \sum_{\substack{s \in W_P \backslash W, [s] = \sigma \\ \mu \in \mathbb{N}\Delta_r(P)}} a^{s\lambda_0 - \mu} q^{\mathcal{L}}_{s,\mu}(P, v \,|\, f)(\log a, \nu, m).$$

The exponents $s\lambda_0 - \mu$ as $s \in W_P \backslash W$, $[s] = \sigma$ and $\mu \in \mathbb{N}\Delta_r(P)$, are all of the form $-\xi$, with $\xi \in -\sigma \cdot \{\lambda_0\} + \mathbb{N}\Delta_r(P)$. Thus, we see that, for $\nu \in \Omega$, $m \in \mathrm{X}_{P,v,+}[R]$ and $a \in A^+_{P\mathrm{q}}(R^{-1})$,

$$(\mathcal{L}_* f)_\nu(mav) = \sum_{\sigma \in W/\sim_{P|Q}} a^{\sigma\nu - \rho_P} \sum_{\xi \in -\sigma \cdot \{\lambda_0\} + \mathbb{N}\Delta_r(P)} a^{-\xi} \, \widetilde{q}_{\sigma,\xi}(\log a, \nu, m) \tag{13.12}$$

with

$$\begin{aligned}\widetilde{q}_{\sigma,\xi} &= \sum_{\substack{s\in W_P\backslash W,\,[s]=\sigma\\ s\lambda_0|_{\mathfrak{a}_{P\mathrm{q}}}+\xi\in\mathbb{N}\Delta_r(P)}} q^{\mathcal{L}}_{s,s\lambda_0|_{\mathfrak{a}_{P\mathrm{q}}}+\xi}(P,v\,|\,f) \qquad (13.13)\\ &\in P_{k+k'}(\mathfrak{a}_{P\mathrm{q}})\otimes\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}},\mathcal{H}',d',C^\infty(\mathrm{X}_{P,v,+}:\tau_P)).\end{aligned}$$

From what we said earlier about the convergence of the series in (13.9), it follows that, for every $R>1$, the inner series on the right-hand side of (13.12) converges neatly on $A^+_{P\mathrm{q}}(R^{-1})$ as a $\Delta_r(P)$-exponential polynomial series with coefficients in the space $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}},\mathcal{H}',d',C^\infty(\mathrm{X}_{P,v,+}[R]:\tau_P))$.

If P is minimal, then $\mathrm{X}_{P,v,+}[R]=\mathrm{X}_{0,v}$ and we see that $\mathcal{L}_*f$ satisfies conditions (b) and (c) of Definition 12.1 with $q_{\sigma,\xi}(P,v\,|\,\mathcal{L}_*f)=\tilde{q}_{\sigma,\xi}$ for $\sigma\in W/\sim_{P|Q}$ $=W/W_Q$. This establishes part (a) of the proposition.

For general P we now see that the functions $\tilde{q}_{\sigma,\xi}$ introduced above coincide with functions $q_{\sigma,\xi}(P,v\,|\,\mathcal{L}_*f)$ introduced in Theorem 7.7. Finally, combining (13.13) and (13.10) we see that we have established part (b) of the proposition as well. $\square$

Lemma 13.5 *Let $\delta\in D_{P_0}$ and $f\in\mathcal{E}^{\mathrm{hyp}}_0(\mathrm{X}_+:\tau:\delta)$. Let $Q\in\mathcal{P}_\sigma$ and $\mathcal{L}\in\mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}},\Sigma_Q)^*_{\mathrm{laur}}$, and put $Y=\operatorname{supp}\mathcal{L}$. There exists a $\delta'\in\mathrm{D}_Q$ such that*

$$\mathcal{L}_*f\in\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+:\tau:\delta').$$

Proof: It follows from Proposition 13.2 that $\mathcal{L}_*f\in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+:\tau)$. Moreover, $\mathrm{reg}_{\mathrm{a}}\mathcal{L}_*f\supset\Omega=\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}\setminus\mathcal{H}_{fQ}(Y)$. Then in view of Definition 12.8 and Remark 12.9 it suffices to establish the existence of a $\delta'\in\mathrm{D}_Q$ such that, for every $\nu\in\Omega$, the function $(\mathcal{L}_*f)_\nu$ is annihilated by the cofinite ideal $I_{\delta',\nu}$.

By linearity we may assume that $\operatorname{supp}\mathcal{L}$ consists of a single point $\lambda_0\in{}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Then $\mathcal{L}=\mathcal{L}_{\lambda_0}$.

Let π_0,u_0 be as in the proof of Lemma 13.4. Then from (13.6) we see that

$$(\mathcal{L}_*f)_\nu(x)=u_0[\pi_0(\,\cdot\,)f(\,\cdot\,+\nu,x)](\lambda_0),$$

for $x\in\mathrm{X}_+$, $\nu\in\Omega$. Moreover, since $(\lambda,x)\mapsto\pi_0(\lambda)f_{\lambda+\nu}(x)$ is smooth in a neighborhood of $\{\lambda_0\}\times\mathrm{X}_+$, it follows that, for $D\in\mathbb{D}(\mathrm{X})$, $\nu\in\Omega$ and $x\in\mathrm{X}_+$,

$$D(\mathcal{L}_*f)_\nu(x)=u_0[\pi_0(\,\cdot\,)D(f_{\,\cdot\,+\nu})(x)](\lambda_0). \qquad (13.14)$$

Put $l=\mathrm{order}(u_0)$ and define $\delta'\in D_Q$ by $\operatorname{supp}\delta'=\{\lambda_0\}+\operatorname{supp}\delta$ and $\delta'(\lambda_0+\Lambda)=\delta(\Lambda)+l$ for $\Lambda\in\operatorname{supp}\delta$. It suffices to prove the following. Let elements $D^\Lambda_i\in\mathbb{D}(\mathrm{X})$ be given for $i=1,\dots,\delta(\Lambda)+l$, for each $\Lambda\in\operatorname{supp}\delta$, and define the differential operator

$$D_\nu\!:=\prod_{\Lambda\in\operatorname{supp}\delta}\prod_{i=1}^{\delta(\Lambda)+l}(D^\Lambda_i-\gamma(D^\Lambda_i,\lambda_0+\Lambda+\nu))\in\mathbb{D}(\mathrm{X}) \qquad (13.15)$$

for $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Then D_ν annihilates $(\mathcal{L}_* f)_\nu$ for each $\nu \in \Omega$.

It follows from (13.4) and (13.14) that

$$D_\nu(\mathcal{L}_* f)_\nu(x) = u_0[\pi_0(\,\cdot\,) D_\nu f_{\,\cdot\,+\nu}(x)](\lambda_0), \tag{13.16}$$

where the dots indicate a variable in ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. We write each factor in D_ν as

$$\begin{aligned} D_i^\Lambda - \gamma(D_i^\Lambda, \lambda_0 + \Lambda + \nu) \\ = [D_i^\Lambda - \gamma(D_i^\Lambda, \,\cdot\, + \Lambda + \nu)] + [\gamma(D_i^\Lambda, \,\cdot\, + \Lambda + \nu) - \gamma(D_i^\Lambda, \lambda_0 + \Lambda + \nu)], \end{aligned}$$

also with variables in ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ indicated by dots. Inserting this into (13.15) and (13.16) we obtain an expression for $D_\nu(\mathcal{L}_* f)_\nu(x)$ as a sum of terms each of the form

$$u_0[\pi_0(\,\cdot\,) \prod_{\Lambda \in \operatorname{supp}\delta} p^\Lambda(\,\cdot\,) D^\Lambda(\,\cdot\,) f_{\,\cdot\,+\nu}(x)](\lambda_0), \tag{13.17}$$

where

$$D^\Lambda(\lambda) = \prod_{i \in S_\Lambda} [D_i^\Lambda - \gamma(D_i^\Lambda, \lambda + \Lambda + \nu)]$$

and

$$p^\Lambda(\lambda) = \prod_{i \in S_\Lambda^c} [\gamma(D_i^\Lambda, \lambda + \Lambda + \nu) - \gamma(D_i^\Lambda, \lambda_0 + \Lambda + \nu)]$$

with S_Λ a subset of $\{1, \dots, \delta(\Lambda) + l\}$ and S_Λ^c its complement in this set. On the one hand, if S_Λ has fewer than $\delta(\Lambda)$ elements for some Λ, there are at least $l+1$ factors in the corresponding product p^Λ. Since each of these factors vanish at λ_0, it follows from the Leibniz rule that then (13.17) vanishes. On the other hand, if for each Λ the set S_Λ has at least $\delta(\Lambda)$ elements, then the differential operator $\prod_\Lambda D^\Lambda(\lambda)$ annihilates $f_{\lambda+\nu}$, again causing (13.17) to vanish. It follows that $D_\nu(\mathcal{L}_* f)_\nu(x) = 0$. $\square$

In the following definition we introduce a notion of asymptotic globality that is somewhat stronger than the one in Definition 8.4. It is motivated by the fact that it carries over by the application of Laurent functionals, as we shall see in Proposition 13.9

Definition 13.6 *Let $Q \in \mathcal{P}_\sigma$, and let $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be finite. Let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$ and $\sigma \in W/\!\sim_{P|Q}$.*

(a) *Let $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be an open subset. A family $f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$ is called holomorphically σ-global along (P, v) if there exists a full open subset Ω^* of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that, for every $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$, the function $\lambda \mapsto q_{\sigma,\xi}(P, v \,|\, f, \,\cdot\,)(\lambda)$ is a holomorphic $P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes C^\infty(\mathrm{X}_{P,v} : \tau_P)$-valued function on $\Omega^* \cap \Omega$, for some $k \in \mathbb{N}$.*

(b) *A family $f \in C^{\mathrm{ep,hyp}}_{Q,Y}(\mathrm{X}_+ : \tau)$ is called holomorphically σ-global along (P, v) if its restriction to $\Omega = \mathrm{reg}_\mathrm{a} f$ is holomorphically σ-global along (P, v), according to (a).*

It is easily seen that the property of holomorphic globality according to (a) of the above definition implies the globality in Definition 8.4. We have the following analogue of Lemma 8.7, describing how the property of holomorphic globality transforms under the action of $N_K(\mathfrak{a}_{\rm q})$.

Lemma 13.7 *Let Q, Y, P, v and σ be as above, and let $f \in C^{\rm ep,hyp}_{Q,Y}({\rm X}_+ : \tau)$.*

If f is holomorphically σ-global along (P, v), then f is holomorphically $u\sigma$-global along (uPu^{-1}, uv), for every $u \in N_K(\mathfrak{a}_{\rm q})$.

Proof: The proof is completely analogous to the proof of Lemma 8.7, involving an application of Lemma 7.10. $\square$

Proposition 13.8 *Let $Q \in \mathcal{P}_\sigma$, $Y \subset {}^*\mathfrak{a}^*_{Q{\rm q}\mathbb{C}}$ a finite subset and let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_{\rm q})$ and $\sigma \in W/\sim_{P|Q}$. Let $f \in C^{\rm ep,hyp}_{Q,Y}({\rm X}_+ : \tau)$ and put $\mathcal{H} = \mathcal{H}_f$, $d = d_f$ and $k = \deg_{\rm a} f$.*

The family f is holomorphically σ-global along (P, v) if and only if, for every element $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$, the function $\lambda \mapsto q_{\sigma,\xi}(P, v \,|\, f, \,\cdot\,)(\lambda)$ belongs to the space $\mathcal{M}(\mathfrak{a}^*_{Q{\rm q}\mathbb{C}}, \mathcal{H}, d, P_k(\mathfrak{a}_{P{\rm q}}) \otimes C^\infty({\rm X}_{P,v} : \tau_P))$.

Proof: The 'if'-statement is obvious. Assume that f is holomorphically σ-global along (P, v), and let $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$. According to Lemma 12.7, the function

$$\lambda \mapsto q_{\sigma,\xi}(P, v \,|\, f, \,\cdot\,, \lambda) \tag{13.18}$$

belongs to the space

$$\mathcal{M}(\mathfrak{a}^*_{Q{\rm q}\mathbb{C}}, \mathcal{H}, d, P_k(\mathfrak{a}_{P{\rm q}}) \otimes C^\infty({\rm X}_{P,v,+} : \tau_P)). \tag{13.19}$$

Let $\Omega = {\rm reg}_{\rm a}(f)$ and let Ω^* be a full open subset of $\mathfrak{a}^*_{Q{\rm q}\mathbb{C}}$ satisfying the properties of Definition 13.6 (a) for the restriction of f to Ω. Then the function (13.18) not only belongs to the space (13.19), but also to the space $\mathcal{O}(\Omega^* \cap \Omega, P_l(\mathfrak{a}_{P{\rm q}}) \otimes C^\infty({\rm X}_{P,v} : \tau_P))$, for some $l \in \mathbb{N}$. In particular we see that this is true with $l = k$.

Let now $X \in \mathfrak{a}_{P{\rm q}}$ be fixed. Then it suffices to show that the function (13.18), with X substituted for the dot, belongs to the space $\mathcal{M}(\mathfrak{a}^*_{Q{\rm q}\mathbb{C}}, \mathcal{H}, d, C^\infty({\rm X}_{P,v} : \tau_P))$. To prove the latter, we fix an arbitrary bounded non-empty open set $\omega \subset \mathfrak{a}^*_{Q{\rm q}\mathbb{C}}$ and put $\pi := \pi_{\omega,d}$, see above Lemma 12.5. Then the function $F : \omega \times {\rm X}_{P,v,+} \to V_\tau$, defined by

$$F(\lambda, m) = \pi(\lambda)\, q_{\sigma,\xi}(P, v \,|\, f, X, \lambda)(m)$$

is C^∞ and holomorphic in its first variable. Moreover, let ω_0 be the full open subset $\omega \cap \Omega^* \cap \Omega$ of ω. Then by what we said above, the restricted function $F|_{\omega_0 \times {\rm X}_{P,v,+}}$ admits a smooth extension to the manifold $\omega_0 \times {\rm X}_{P,v}$. It now follows from Corollary 18.2 that F has a unique smooth extension to $\omega \times {\rm X}_{P,v}$; this extension is holomorphic in the first variable. It follows that the function $\lambda \mapsto \pi(\lambda) q_{\sigma,\xi}(P, v \,|\, f, X, \lambda)$ belongs to $\mathcal{O}(\omega, C^\infty({\rm X}_{P,v} : \tau_P))$. Since ω was arbitrary, this completes the proof. $\square$

Proposition 13.9 *Let $f \in C_0^{\rm ep,hyp}(\mathrm{X}_+ : \tau)$, let $Q \in \mathcal{P}_\sigma$ and let $\mathcal{L}$ be a Laurent functional in $\mathcal{M}({}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \Sigma_Q)^*_{\rm laur}$. Put $Y = \operatorname{supp} \mathcal{L}$. Let $P \in \mathcal{P}_\sigma$, $v \in N_K(\mathfrak{a}_\mathrm{q})$ and $\sigma \in W/\sim_{P|Q}$.*

If f is holomorphically s-global along (P, v) for every $s \in W_P \backslash W$ with $[s] = \sigma$, then $\mathcal{L}_* f \in C_{Q,Y}^{\rm ep,hyp}(\mathrm{X}_+ : \tau)$ is holomorphically σ-global along (P, v).

Proof: It follows from Proposition 13.2 (a) that $\mathcal{L}_* f \in C_{Q,Y}^{\rm ep,hyp}(\mathrm{X}_+ : \tau)$. Assume that f satisfies the globality assumptions. Then it remains to establish the assertion on σ-globality for $\mathcal{L}_* f$.

Let $k = \deg_\mathrm{a} f$. Let $\mathcal{H} = \mathcal{H}_f$, $d = d_f$ and $\mathcal{H}' = \mathcal{H}_Q(Y)$. Moreover, let $d' \colon \mathcal{H}' \to \mathbb{N}$ be associated with these data as in Lemma 13.1 and let $k' \in \mathbb{N}$ be associated as in Proposition 13.2 (a). According to the latter proposition, the set $\Omega' = \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}'$ is contained in $\mathrm{reg}_\mathrm{a}(\mathcal{L}_* f)$.

Let $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$. Moreover, let $s \in W_P \backslash W$ be such that $[s] = \sigma$ and let $\lambda_0 \in Y$ be such that $\eta \colon = s\lambda_0|_{\mathfrak{a}_{P\mathrm{q}}} + \xi$ belongs to $\mathbb{N}\Delta_r(P)$. Then by Proposition 13.8, the function

$$\lambda \mapsto q_{s,\eta}(P, v \,|\, f, \,\cdot\,, \lambda)$$

belongs to $\mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, P_k(\mathfrak{a}_{P\mathrm{q}}) \otimes C^\infty(\mathrm{X}_{P,v} : \tau_P))$. Using Lemma 13.4 with $C^\infty(\mathrm{X}_{P,v} : \tau_P)$ in place of U, we see that, for $X \in \mathfrak{a}_{P\mathrm{q}}$, the function

$$\varphi_X \colon = \mathcal{L}_{\lambda_0 *}^{P,s}[q_{s,\eta}(P, v \,|\, f, X, \,\cdot\,)]$$

belongs to $\mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}', d', P_{k'}(\mathfrak{a}_{P\mathrm{q}}) \otimes C^\infty(\mathrm{X}_{P,v} : \tau_P))$. Moreover, it depends on $X \in \mathfrak{a}_{P\mathrm{q}}$ as a polynomial function of degree at most k. It follows that the function $(\nu, X) \mapsto \varphi_X(\nu)(X)$ belongs to the space

$$\mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \mathcal{H}', d', P_{k+k'}(\mathfrak{a}_{P\mathrm{q}}) \otimes C^\infty(\mathrm{X}_{P,v} : \tau_P)). \tag{13.20}$$

Each term in the finite sum (13.5) is of this form. Hence the function

$$(\nu, X) \mapsto q_{\sigma,\xi}(P, v \,|\, \mathcal{L}_* f, X, \nu)$$

belongs to the space (13.20) as well. This holds for all $\xi \in -\sigma \cdot Y + \mathbb{N}\Delta_r(P)$. Therefore the restriction of $\mathcal{L}_* f$ to $\mathrm{reg}_\mathrm{a}(\mathcal{L}_* f)$ satisfies Definition 13.6 (a) with $\Omega^* = \Omega'$. $\square$

The following definition is an analogue of the final part of Definition 12.8, replacing the globality condition by a condition of holomorphic globality.

Definition 13.10 *Let $Q \in \mathcal{P}_\sigma$ and let $\delta \in \mathrm{D}_Q$. We define*

$$\mathcal{E}_Q^{\rm hyp}(\mathrm{X}_+ : \tau : \delta)_{\rm hglob}$$

to be the space of functions $f \in \mathcal{E}_Q^{\rm hyp}(\mathrm{X}_+ : \tau : \delta)$ satisfying the following condition.

For each $s \in W$ and every $P \in \mathcal{P}_\sigma^1$ with $s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{P\mathrm{q}}$, the family f is holomorphically $[s]$-global along (P, v), for all $v \in N_K(\mathfrak{a}_\mathrm{q})$; here $[s]$ denotes the image of s in $W/\!\sim_{P|Q} = W_P\backslash W/W_Q$.

If $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a finite subset, we define

$$\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{hglob}} = \mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta) \cap \mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{hglob}}.$$

It is easily seen that $\mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{hglob}} \subset \mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}$. As in Lemma 9.7 the above condition allows a reduction to a smaller set of (s, P).

Lemma 13.11 *Let $Q \in \mathcal{P}_\sigma$ be standard, let $\delta \in \mathrm{D}_Q$ and $f \in \mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)$. Then f belongs to $\mathcal{E}^{\mathrm{hyp}}_{Q}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{hglob}}$ if and only if the following condition is fulfilled.*

For each $s \in W$ and every $\alpha \in \Delta$ with $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$, the family f is holomorphically $[s]$-global along (P_α, v), for all $v \in N_K(\mathfrak{a}_\mathrm{q})$.

Proof: The proof is similar to the proof of Lemma 9.7, involving Lemma 13.7 instead of Lemma 8.7. $\square$

We now come to the main result of this section, which provides a source of functions to which the vanishing theorem (Theorem 12.10) can be applied.

Theorem 13.12 *Let $\delta \in D_{P_0}$ and $f \in \mathcal{E}^{\mathrm{hyp}}_{0}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{hglob}}$, let $Q \in \mathcal{P}_\sigma$ be a standard σ-parabolic subgroup and let $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$. Put $Y = \operatorname{supp} \mathcal{L}$. Then there exists a $\delta' \in \mathrm{D}_Q$ such that*

$$\mathcal{L}_* f \in \mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta')_{\mathrm{hglob}}.$$

Proof: From Lemma 13.5 it follows that $\mathcal{L}_* f$ is a family in $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta')$ for some $\delta' \in \mathrm{D}_Q$. Let $s \in W$ and $\alpha \in \Delta$ be such that $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$. Then every $t \in W_\alpha s W_Q$ also satisfies the condition $t^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$; hence $t(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{\alpha\mathrm{q}}$. Thus, from the hypothesis it follows that f is holomorphically $W_\alpha t$-global along (P_α, v) for every t in the double coset $W_\alpha s W_Q$. According to Lemma 6.5, see also Remark 9.6, the latter set equals the class $[s]$ of s for the equivalence relation $\sim_{P_\alpha|Q}$ in W. It now follows from Proposition 13.9 that $\mathcal{L}_* f$ is holomorphically $[s]$-global along (P_α, v). We conclude that $\mathcal{L}_* f$ satisfies the conditions of Lemma 13.11, hence belongs to $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta')_{\mathrm{hglob}}$. $\square$

14 Partial Eisenstein integrals

In this section we will define partial Eisenstein integrals and show that they belong to the families of eigenfunctions introduced in the previous section.

We start by recalling some properties of Eisenstein integrals. Let $P \in \mathcal{P}_\sigma^{\min}$ be a minimal σ-parabolic subgroup. Let (τ, V_τ) be a finite dimensional unitary representation of K. Let $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ be a fixed set of representatives for $W/W_{K\cap H}$. Following [9], eq. (5.1), we define the complex linear space ${}^\circ\mathcal{C} = {}^\circ\mathcal{C}(\tau)$ as the following formal direct sum of finite dimensional linear spaces

$$ {}^\circ\mathcal{C} := \oplus_{w\in\mathcal{W}} \; C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}}). \tag{14.1} $$

Every summand in the above sum, as $w \in \mathcal{W}$, is a finite dimensional subspace of the Hilbert space $L^2(\mathrm{X}_{0,w}, V_\tau)$; here the L^2-inner product is defined relative to the normalized M-invariant measure of the compact space $\mathrm{X}_{0,w} = M/M \cap wHw^{-1}$ and the Hilbert structure of V_τ. Thus, every summand is a finite dimensional Hilbert space of its own right. The formal direct sum ${}^\circ\mathcal{C}$ is equipped with the direct sum inner product, turning (14.1) into an orthogonal direct sum.

For $\psi \in {}^\circ\mathcal{C}$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $x \in \mathrm{X}$, the Eisenstein integral $E(\psi : \lambda : x) = E(P : \psi : \lambda : x)$ and its normalized version $E^\circ(\psi : \lambda : x) = E^\circ(P : \psi : \lambda : x)$ are defined as in [9], § 5. The Eisenstein integrals are τ-spherical functions of x, depend meromorphically on λ and linearly on ψ. We view $E^\circ(\lambda : x) := E^\circ(\,\cdot\, : \lambda : x)$ (and similarly its unnormalized version) as an element of $\mathrm{Hom}({}^\circ\mathcal{C}, V_\tau) \simeq V_\tau \otimes {}^\circ\mathcal{C}^*$. Thus, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, $E^\circ(\lambda)$ is a $\tau \otimes 1$-spherical function on X. The connection between the unnormalized and the normalized Eisenstein integral is now given by the identity

$$ E^\circ(\lambda : x) = E(\lambda : x) \circ C(1 : \lambda)^{-1}, \qquad (x \in \mathrm{X}), \tag{14.2} $$

for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Here $C(1 : \lambda) := C_{P|P}(1 : \lambda)$ is a meromorphic $\mathrm{End}({}^\circ\mathcal{C})$-valued function of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$; see [9], p. 283.

The Eisenstein integral is $\mathbb{D}(\mathrm{X})$-finite. In fact, we recall from [9], eq. (5.11), that there exists a homomorphism μ from $\mathbb{D}(\mathrm{X})$ to the algebra of $\mathrm{End}({}^\circ\mathcal{C})$-valued polynomial functions on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ such that

$$ DE^\circ(\lambda) = [I \otimes \mu(D : \lambda)^*]E^\circ(\lambda), \qquad (D \in \mathbb{D}(\mathrm{X})). $$

It now follows from Lemma 5.3 that, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, the Eisenstein integral $E^\circ(\lambda)$ belongs to $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau \otimes 1)$. It therefore has expansions of the form (2.12). These expansions have been determined explicitly in [8]. We recall some of the results of that paper.

In [8], eq. (15), we define a function $\Phi_P(\lambda : \,\cdot\,)$ on $A_{\mathrm{q}}^+(P)$ by an exponential polynomial series with coefficients in $\mathrm{End}(V_\tau^{M\cap K\cap H})$ of the form

$$ \Phi_P(\lambda : a) = a^{\lambda - \rho_P} \sum_{\nu \in \Delta(P)} a^{-\nu} \Gamma_{P,\nu}(\lambda), \qquad (a \in A_{\mathrm{q}}^+(P)). \tag{14.3} $$

Note that here P replaces the Q of [8], Sect. 5; also, in [8] we suppressed the Q in the notation. The coefficients in the expansion (14.3) are defined by recursive relations (see [8], eq. (18) and Prop. 5.2); it follows from these that

the coefficients depend meromorphically on λ, and that the expansion (14.3) converges to a smooth function on $A_{\mathrm{q}}^+(P)$, depending meromorphically on λ. In fact, we have the following stronger result.

Let $\Pi_{\Sigma,\mathbb{R}}$ be the collection of polynomial functions $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$ that can be written as finite products of linear factors of the form $\lambda \mapsto \langle \lambda\,,\,\alpha\rangle - c$, with $\alpha \in \Sigma$ and $c \in \mathbb{R}$. For $R \in \mathbb{R}$, we define the set

$$\mathfrak{a}_{\mathrm{q}}^*(P,R) := \{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \mid \mathrm{Re}\,\langle \lambda\,,\,\alpha\rangle < R \;\; \forall \alpha \in \Sigma(P)\}.$$

Lemma 14.1 *Let $R \in \mathbb{R}$. Then there exists a polynomial function $p \in \Pi_{\Sigma,\mathbb{R}}$ such that the functions $p\Gamma_{P,\nu}$, for $\nu \in \mathbb{N}\Delta(P)$, are all regular on $\mathfrak{a}_{\mathrm{q}}(P,R)$. Moreover, if p is a polynomial function with the above property, then the series*

$$\sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} p(\,\cdot\,)\Gamma_{P,\nu}(\,\cdot\,) \tag{14.4}$$

converges neatly on $A_{\mathrm{q}}^+(P)$ as an exponential series with coefficients in $\mathcal{O}(\mathfrak{a}_{\mathrm{q}}^*(P,R)) \otimes \mathrm{End}(V_\tau^{M\cap K\cap H})$. In particular, the function $(a,\lambda) \mapsto p(\lambda)\Phi_P(\lambda : a)$ is smooth on $A_{\mathrm{q}}^+(P) \times \mathfrak{a}_{\mathrm{q}}^*(P,R)$, and in addition holomorphic in its second variable.

Proof: Let p_R be the polynomial function described in [8], Thm. 9.1. As in the proof of that theorem, it follows from the estimates in [8], Thm. 7.4, that the power series

$$\Psi(\lambda : z) = \sum_{\nu \in \mathbb{N}\Delta(P)} z^{-\nu} p_R(\lambda)\Gamma_{P,\nu}(\lambda)$$

converges absolutely locally uniformly in the variables $z \in D^{\Delta(P)}$ and $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(P,R)$. Here we have used the notation of Sect. 1 of the present paper. Since $p_R(\lambda)\Phi_P(\lambda : a) = a^{\lambda-\rho_P}\Psi(\lambda : \underline{z}(a))$, for $a \in A_{\mathrm{q}}^+(P)$, this implies all assertions of the lemma with p_R in place of p.

This is not immediately good enough, since p_R is a finite product of linear factors of the form $\lambda \mapsto \langle \lambda\,,\,\nu\rangle - c$, with $\nu \in \mathbb{N}\Delta(P)$ and $c \in \mathbb{R}$, see [8], the equation preceding Lemma 7.3. To overcome this, we invoke [8], Prop. 9.4. It follows from that result and its proof that there exists a $p \in \Pi_{\Sigma,\mathbb{R}}$ such that $p\Gamma_{P,\nu}$ is regular on $\mathfrak{a}_{\mathrm{q}}^*(P,R)$, for every $\nu \in \mathbb{N}\Delta(P)$. Let p be any polynomial with this property, and let ${}^{\backprime}p$ be the least common multiple of p and p_R. Then all assertions of the lemma hold with ${}^{\backprime}p$ in place of p. Let q be the quotient of ${}^{\backprime}p$ by p. Denote the image of the linear endomorphism $m_q \colon \varphi \mapsto q\varphi$ of $\mathcal{O}(\mathfrak{a}_{\mathrm{q}}^*(P,R))$ by $\mathcal{F}$, and equip this space with the locally convex topology induced from $\mathcal{O}(\mathfrak{a}_{\mathrm{q}}^*(P,R))$. It follows from an easy application of the Cauchy integral formula that m_q is a topological linear isomorphism from $\mathcal{O}(\mathfrak{a}_{\mathrm{q}}^*(P,R))$ onto $\mathcal{F}$; see also [9], Lemma 20.7. As said above, all assertions of the lemma hold with ${}^{\backprime}p$ in place of p; on the other hand, by the hypothesis the series (14.4) with ${}^{\backprime}p$ in place of p has coefficients in $\mathcal{F}$. Applying the continuous linear map m_q^{-1} to that series, we infer that all assertions of the lemma are true with the polynomial $q^{-1}{}^{\backprime}p = p$. $\square$

Following [8], Sect. 11, we define the function $\Phi_{P,w} \colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \times A_{\mathrm{q}}^+(P) \to \mathrm{End}(V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}})$, for $w \in \mathcal{W}$, by

$$\Phi_{P,w}(\lambda \colon a) = \tau(w) \circ \Phi_{w^{-1}Pw}(w^{-1}\lambda \colon w^{-1}aw) \circ \tau(w)^{-1}. \tag{14.5}$$

Following [9], p. 283, we define normalized C-functions $C^\circ(s \colon \lambda) = C^\circ_{P|P}(s \colon \lambda)$, for $s \in W$, by

$$C^\circ(s \colon \lambda) = C(s \colon \lambda) \circ C(1 \colon \lambda)^{-1}; \tag{14.6}$$

these are $\mathrm{End}({}^\circ\mathcal{C})$-valued meromorphic functions of $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. From (14.2) and [8], eq. (54), we now obtain the following description of the normalized Eisenstein integral in terms of the functions $\Phi_{P,w}$. Let $\psi \in {}^\circ\mathcal{C}$ and $w \in \mathcal{W}$. Then, for $a \in A_{\mathrm{q}}^+(P)$,

$$E^\circ(\lambda \colon aw)\psi = \sum_{s \in W} \Phi_{P,w}(s\lambda \colon a)[C^\circ(s \colon \lambda)\psi]_w(e), \tag{14.7}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}_{\mathrm{q}}^*$.

From (14.5) and (14.3) it follows that, for $w \in \mathcal{W}$, the function $\Phi_{P,w}$ is given by the series

$$\Phi_{P,w}(\lambda \colon a) = a^{\lambda - \rho_P} \sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \Gamma_{P,w,\nu}(\lambda), \tag{14.8}$$

with coefficients

$$\Gamma_{P,w,\nu}(\lambda) = \tau(w) \circ \Gamma_{w^{-1}Pw, w^{-1}\nu}(w^{-1}\lambda) \circ \tau(w)^{-1}. \tag{14.9}$$

We now have the following result on the convergence of the series (14.8).

Corollary 14.2 *Let $w \in \mathcal{W}$. Then there exists a locally finite real Σ-hyperplane configuration $\mathcal{H} = \mathcal{H}_w$ in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and a map $d = d_w \colon \mathcal{H} \to \mathbb{N}$, such that the functions $\Gamma_{P,w,\nu}$ belong to $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, \mathrm{End}(V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}))$, for every $\nu \in \mathbb{N}\Delta(P)$. Moreover, the series*

$$\sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \Gamma_{P,w,\nu} \tag{14.10}$$

converges neatly on $A_{\mathrm{q}}^+(P)$ as an exponential polynomial series with coefficients in the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, \mathrm{End}(V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}))$. In particular, the function $\lambda \mapsto \Phi_{P,w}(\lambda \colon \cdot\,)$ belongs to the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(A_{\mathrm{q}}^+(P)) \otimes \mathrm{End}(V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}))$.

Proof: For $w = 1$ the assertion of the corollary follows immediately from Lemma 14.1. For arbitrary $w \in \mathcal{W}$ it then follows by application of (14.9). $\square$

For $s \in W$ we define the so called partial Eisenstein integral $E_{+,s}(\lambda) = E_{+,s}(P : \lambda)$ as the $\tau \otimes 1$-spherical function $\mathrm{X}_+ \to V_\tau \otimes {}^\circ\mathcal{C}^*$ determined by

$$E_{+,s}(\lambda : aw)\psi = \Phi_{P,w}(s\lambda : a)[C^\circ(s : \lambda)\psi]_w(e), \tag{14.11}$$

for $\psi \in {}^\circ\mathcal{C},\ w \in \mathcal{W},\ a \in A_\mathrm{q}^+(P)$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ (use the isomorphism (2.8)). It follows from Corollary 14.2 that $E_{+,s}$ is a meromorphic $C^\infty(\mathrm{X}_+ : \tau\otimes 1)$-valued function on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. By sphericality it follows from (14.7) and (14.11) that

$$E^\circ(\lambda) = \sum_{s\in W} E_{+,s}(\lambda) \qquad \text{on} \qquad \mathrm{X}_+. \tag{14.12}$$

It follows from the definitions and the isomorphism (2.9) that, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, the function $E_{+,s}(\lambda)\psi$ belongs to $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau\otimes 1)$ for each $\psi \in {}^\circ\mathcal{C}$. Moreover,

$$\mathrm{Exp}(P, v \,|\, E_{+,s}(\lambda)\psi) \subset s\lambda - \rho_P - \mathbb{N}\Delta(P), \tag{14.13}$$

for every $v \in \mathcal{W}$ and hence also for every $v \in N_K(\mathfrak{a}_\mathrm{q})$. Thus, we see that (14.12) is the splitting of Lemma 2.2 applied to the Eisenstein integral. We abbreviate $E_+(\lambda) = E_{+,1}(\lambda)$. Then from (14.11) and (14.6) we see that

$$E_+(\lambda)(aw)\psi = \Phi_{P,w}(\lambda : a)\psi_w(e),$$

for $\psi \in {}^\circ\mathcal{C},\ w \in \mathcal{W},\ a \in A_\mathrm{q}^+(P)$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Moreover, the following holds as a meromorphic identity in $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$

$$E_{+,s}(\lambda : x) = E_+(s\lambda : x)\, C^\circ(s : \lambda). \tag{14.14}$$

In the next lemma we will need the following notation. If $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$, we denote by ${}^\circ\mathcal{C}[\Lambda]$ the subspace of ${}^\circ\mathcal{C}$ consisting of elements ψ satisfying $\mu(D : \lambda)\psi = \gamma(D : \Lambda + \lambda)\psi$ for all $D \in \mathbb{D}(\mathrm{X})$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. We recall from [9], eq. (5.14), that ${}^\circ\mathcal{C}$ is a finite direct sum

$${}^\circ\mathcal{C} = \oplus_\Lambda\, {}^\circ\mathcal{C}[\Lambda],$$

where Λ ranges over a finite subset L_τ of $\mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$. For each $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$, we denote by $\mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$ the space $\mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \delta)$ (see Definition 12.8) where $\delta \in D_P$ is the characteristic function of $\{\Lambda\}$.

Lemma 14.3 *Let $P \in \mathcal{P}_\sigma^{\min}$, $t \in W$ and $\psi \in {}^\circ\mathcal{C}[\Lambda]$, where $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$. Define the family $f = f_{\{t\}} \colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \times \mathrm{X}_+ \to V_\tau$, by*

$$f(\lambda, x) = E_{+,t}(P : \lambda : x)\psi.$$

Then $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$ and $\deg_\mathrm{a} f = 0$.

Proof: According to Definition 12.8 and Remark 12.9, in order to prove that $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$ we must establish that $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$ and that f_λ is annihilated by $I_{\Lambda+\lambda}$ for λ in a non-empty open subset of $\mathrm{reg}_\mathrm{a} f$.

We first assume that $t = 1$. Then $f(\lambda, x) = E_+(\lambda : x)\psi$. It follows immediately from [8] Cor. 9.3 and the hypothesis on ψ that f_λ is annihilated by $I_{\Lambda+\lambda}$ for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. We will now show that $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$. Let $\mathcal{H}$ be the union of the hyperplane configurations $\mathcal{H}_w$, $w \in \mathcal{W}$, of Corollary 14.2, and let $d: \mathcal{H} \to \mathbb{N}$ be defined by $d = \max_{w \in \mathcal{W}} d_w$ (see Remark 11.1). Then for every complete locally convex space U, the spaces $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}_w, d_w, U)$ are included in the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, U)$, with continuous inclusion maps. Hence for each $w \in \mathcal{W}$ the series (14.10) converges as a $\Delta(P)$-exponential polynomial series on $A_{\mathrm{q}}^+(P)$, with coefficients in the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, \mathrm{End}(V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}))$. Moreover, the function $\lambda \mapsto \Phi_{P,w}(\lambda : \cdot\,)$ is contained in $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(A_{\mathrm{q}}^+(P)) \otimes \mathrm{End}(V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}))$.

On the other hand, from (14.11) and (14.6) with $s = 1$, it follows that

$$T_{P,w}^{\downarrow}(f_\lambda)(a) = f(\lambda, aw) = \Phi_{P,w}(\lambda : a)\psi_w(e), \tag{14.15}$$

for all $w \in \mathcal{W}$, $a \in A_{\mathrm{q}}^+(P)$ and $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}$. Hence the function $\lambda \mapsto T_{P,w}^{\downarrow}(f_\lambda)$ belongs to the space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(A_{\mathrm{q}}^+(P), V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}))$. In view of the isomorphism (2.8), it now follows that the function $\lambda \mapsto f_\lambda$ belongs to $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(\mathrm{X}_+ : \tau))$. This establishes condition (a) of Definition 12.1, with $Q = P_0$ and $Y = \{0\}$.

The evaluation map $\psi \mapsto \psi(e)$ is a linear isomorphism from $C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}})$ onto $V_\tau^{K_{\mathrm{M}} \cap wHw^{-1}}$. Thus, for $w \in \mathcal{W}$ and $\nu \in \mathbb{N}\Delta(P)$ we may define a function $\tilde{q}_{1,\nu}(P, w \,|\, f) : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}})$ by

$$\tilde{q}_{1,\nu}(P, w \,|\, f, \lambda, e) = \Gamma_{P,w,\nu}(\lambda)\psi_w(e), \tag{14.16}$$

for $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. Then $\tilde{q}_{1,\nu}(P, w \,|\, f) \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d, C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}}))$. Moreover, from what we said earlier about the convergence of the series (14.10), it follows that, for $w \in \mathcal{W}$, the series

$$\sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \tilde{q}_{1,\nu}(P, w \,|\, f)$$

converges neatly as a $\Delta(P)$-exponential polynomial series on $A_{\mathrm{q}}^+(P)$, with coefficients in $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(\mathrm{X}_{0,w} : \tau_{\mathrm{M}}))$.

From (14.15), (14.8) and (14.16) it follows by sphericality that, for $w \in \mathcal{W}$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}$, $m \in \mathrm{X}_{0,w}$ and $a \in A_{\mathrm{q}}^+(P)$,

$$f_\lambda(maw) = a^{\lambda - \rho_P} \sum_{\nu \in \mathbb{N}\Delta(P)} a^{-\nu} \tilde{q}_{1,\nu}(P, w \,|\, f)(\lambda, m),$$

This establishes assertions (b) and (c) of Definition 12.1 with a fixed P, arbitrary $v \in \mathcal{W}$, and, for $\nu \in \mathbb{N}\Delta(P)$, $X \in \mathfrak{a}_{\mathrm{q}}$,

$$q_{s,\nu}(P, v \,|\, f, X) = \begin{cases} \tilde{q}_{1,\nu}(P, v \,|\, f) & \text{for} \quad s = 1; \\ 0 & \text{for} \quad s \in W \setminus \{1\}. \end{cases}$$

In view of Remark 12.3 we have shown that $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$. Moreover, $\deg_{\mathrm{a}} f = 0$. This completes the proof for $t = 1$.

Let now $t \in W$ be arbitrary and let $\tilde{t} \in W_0(\mathfrak{b})$ be such that $\tilde{t}|_{\mathfrak{a}_q} = t$; see the text preceding Lemma 5.5. From (14.14) we see that

$$f(\lambda, x) = E_+(t\lambda : x) C^\circ(t : \lambda)\psi. \tag{14.17}$$

It follows from [9], Lemma 20.6, that there exists a Σ-configuration $\mathcal{H}'$ in $\mathfrak{a}_{q\mathbb{C}}^*$ and a map $d' : \mathcal{H}' \to \mathbb{N}$, such that

$$C^\circ(t : \cdot\,) \in \mathcal{M}(\mathfrak{a}_{q\mathbb{C}}^*, \mathcal{H}', d', \mathrm{End}(^\circ\mathcal{C})). \tag{14.18}$$

From [9], eq. (5.13), it follows that $C^\circ(t : \lambda)$ maps $^\circ\mathcal{C}[\Lambda]$ into $^\circ\mathcal{C}[\tilde{t}\Lambda]$. Fix a basis $\psi_1, \dots, \psi_s$ for $^\circ\mathcal{C}[\tilde{t}\Lambda]$. Then there exist unique functions $c_j \in \mathcal{M}(\mathfrak{a}_{q\mathbb{C}}^*, \mathcal{H}', d')$ such that

$$C^\circ(t : \lambda)\psi = \sum_{j=1}^{r} c_j(\lambda)\psi_j. \tag{14.19}$$

For $1 \le j \le r$ we define the family $g_j : \mathfrak{a}_{q\mathbb{C}}^* \times \mathrm{X}_+ \to V_\tau$ by

$$g_j(\lambda, x) = E_+(\lambda : x)\psi_j. \tag{14.20}$$

Then by the first part of the proof, each g_j belongs to $\mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \tilde{t}\Lambda)$. Moreover, for every $1 \le j \le r$, the family g_j satisfies the conditions of Definition 12.1 with $Q = P_0$ and $Y = \{0\}$, with $\mathcal{H}$ and d as in the first part of the proof, and with $k = 0$.

For $1 \le j \le r$ we define the family $f_j : \mathfrak{a}_{q\mathbb{C}}^* \times \mathrm{X}_+ \to V_\tau$ by $f_j(\lambda, x) = g_j(t\lambda, x)$. Then we readily see that f_j satisfies the conditions of Definition 12.1 with $t^{-1}\mathcal{H}$ and $d \circ t$ in place of $\mathcal{H}$ and d, respectively, and with $k = 0$. Hence $f_j \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$. Since $I_{\tilde{t}\Lambda + t\lambda} = I_{\Lambda + \lambda}$ we see that $f_j \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$. Moreover, $\deg_{\mathrm{a}} f_j = 0$.

Combining (14.17) and (14.19) with (14.20) and the definition of f_j, we find that

$$f(\lambda, x) = \sum_{j=1}^{r} c_j(\lambda) f_j(\lambda, x).$$

Let $\mathcal{H}'' = t^{-1}\mathcal{H} \cup \mathcal{H}'$ and define $d'' : \mathcal{H}'' \to \mathbb{N}$ by $d''(H) = d(tH) + d'(H)$ (see Remark 11.1). Then by linearity it readily follows that f satisfies all conditions of Definition 12.1, with $k = 0$ and with $\mathcal{H}''$ and d'' in place of $\mathcal{H}$ and d, respectively. Hence $f \in C_0^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \tau)$ and $\deg_{\mathrm{a}} f = 0$. Moreover, for generic λ, f_λ is annihilated by $I_{\Lambda+\lambda}$, and hence $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$. $\square$

Corollary 14.4 *Let assumptions be as in Lemma 14.3 and let Q be a σ-parabolic subgroup. Let $\mathcal{L} \in \mathcal{M}(^*\mathfrak{a}_{Qq\mathbb{C}}^*, \Sigma_Q)_{\mathrm{laur}}^*$. Then $\mathcal{L}_* f \in \mathcal{E}_{Q,Y}^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \delta)$ for $Y = \mathrm{supp}\,\mathcal{L}$ and δ a suitable element in D_Q. Moreover,*

$$\mathrm{Exp}(P, v \,|\, (\mathcal{L}_* f)_\nu) \subset t(\nu + Y) - \rho_P - \mathbb{N}\Delta(P)$$

for $v \in N_K(\mathfrak{a}_q)$ and $\nu \in \mathrm{reg}_{\mathrm{a}} \mathcal{L}_* f$.

Proof: This follows immediately from Lemmas 14.3 and 13.5, and from (14.13) combined with the final statement in Proposition 13.2 (b). $\square$

Lemma 14.5 *Let $\psi \in {}^\circ\mathcal{C}[\Lambda]$ where $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$. Then the family $f\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \times \mathrm{X}_+ \to V_\tau$, defined by*

$$f(\lambda, x) = E^\circ(P_0 : \lambda : x)\psi$$

belongs to $\mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$. Moreover, $\deg_{\mathrm{a}} f = 0$ and for all $P \in \mathcal{P}_\sigma, v \in N_K(\mathfrak{a}_{\mathrm{q}})$ and every $s \in W_P\backslash W$, the family f is holomorphically s-global along (P, v).

Proof: The function f equals the sum, for $t \in W$, of the functions $f_{\{t\}}$ defined in Lemma 14.3, with P_0 in place of P. Hence $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$ and $\deg_{\mathrm{a}} f = 0$. Moreover, for each $\lambda \in \mathrm{reg}_{\mathrm{a}} f$, the function f_λ is asymptotically global along all pairs (P, v) by Proposition 8.8. Thus, it remains to prove the assertion on holomorphic globality. In view of Lemma 13.7, it suffices to do this for arbitrary $P \in \mathcal{P}_\sigma$ and the special value $v = e$.

In the rest of this proof we shall use notation of the paper [7]. According to [7], Lemma 14, there exists a locally finite collection $\mathcal{H}$ of Σ-hyperplanes such that $\lambda \mapsto f_\lambda$ is holomorphic on $\Omega_0 := \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \setminus \cup\mathcal{H}$, with values in $C^\infty(\mathrm{X} : \tau)$. According to the same mentioned lemma it follows that $f \in \mathcal{E}_*(G/H, V_\tau, \Omega_0)$. According to [7], p. 562, Cor. 1, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ the function f_λ has an asymptotic expansion of the form

$$f_\lambda(x \exp tX) \sim \sum_{\substack{s \in W_P\backslash W \\ \nu \in \mathbb{N}\Delta_r(P)}} p_{P,\nu}(f_\lambda : s : \lambda)(x)\, e^{(s\lambda - \rho_P - \nu)(tX)} \quad (t \to \infty) \qquad (14.21)$$

for $X \in \mathfrak{a}_{P\mathrm{q}}$ at every $X_0 \in \mathfrak{a}_{P\mathrm{q}}^+$. Proposition 10 of [7] is valid with $\mathcal{E}_*(G/H, V_\tau, \Omega_0)$ in place of $\mathcal{E}_*(G/H, \Lambda, \Omega_0)$, by the remarks in the beginning of [7], Sect. 12. In particular, there exists a full open subset ${}''\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ such that, for all $s \in W_P\backslash W$ and $\nu \in \mathbb{N}\Delta_r(P)$, the coefficient $p_{P,\nu}(f_\lambda : s : \lambda)$ is holomorphic as a $C^\infty(G, V_\tau)$-valued function of λ on the full open set $\Omega_0 \cap {}''\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

On the other hand, since $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$, and $\deg_{\mathrm{a}} f = 0$, the expansion (12.2) holds, with $k = 0$ and $Y = \{0\}$, for all $\lambda \in \Omega := \mathrm{reg}_{\mathrm{a}} f$. Thus, if $\lambda \in \Omega \cap \mathfrak{a}_{\mathrm{q}\mathbb{C}}^{*0}(P, \{0\})$ is generic, then it follows from comparing the expansions (14.21) and (12.2), and using Lemma 6.2 and uniqueness of asymptotics (see the proof of Lemma 1.7), that

$$q_{s,\nu}(P, e \,|\, f, X, \lambda)(m) = p_{P,\nu}(f_\lambda : s : \lambda)(m), \qquad (14.22)$$

for all $s \in W_P\backslash W$, $\nu \in \mathbb{N}\Delta_r(P)$, $X \in \mathfrak{a}_{P\mathrm{q}}$ and $m \in M_{P,+}$; here we have written $M_{P,+}$ for the preimage of $\mathrm{X}_{P,e,+}$ in M_P.

By analytic continuation the equality (14.22) holds for all λ in the full, hence connected, open subset $\Omega' := \Omega \cap \Omega_0 \cap {}''\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. In particular it follows that $\lambda \mapsto q_{s,\nu}(P, e \,|\, f, \lambda)$ is holomorphic on Ω' as a function with values in $P_0(\mathfrak{a}_{P\mathrm{q}}) \otimes C^\infty(\mathrm{X}_{P,e} : \tau_P)$, for all $s \in W_P\backslash W$ and $\nu \in \mathbb{N}\Delta_r(P)$. This establishes the assertion on holomorphic globality, see Definition 13.6. $\square$

Lemma 14.6 *Let $\Lambda \in \mathfrak{b}_{\mathrm{k}\mathbb{C}}^*$, $\psi \in {}^\circ\mathcal{C}[\Lambda]$, $S \subset W$ and define $f_S\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \times \mathrm{X}_+ \to V_\tau$ by*

$$f_S(\lambda, x) := \sum_{s \in S} E_{+,s}(P_0 : \lambda : x)\psi.$$

Then the family f_S belongs to $\mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$. Moreover, let $t \in W$ and $\alpha \in \Delta$, and assume that either $W_\alpha t \subset S$ or $W_\alpha t \cap S = \emptyset$, where $W_\alpha = \{1, s_\alpha\}$. Then the family f_S is holomorphically $W_\alpha t$-global along (P_α, v), for every $v \in N_K(\mathfrak{a}_{\mathrm{q}})$.

Proof: The first assertion is an immediate consequence of Lemma 14.3 with P_0 in place of P.

Let $v \in N_K(\mathfrak{a}_{\mathrm{q}})$. It follows from (14.13) and Theorem 3.5 that

$$\mathrm{Exp}(P_\alpha, v \,|\, f_{\{s\}\lambda}) \subset s\lambda|_{\mathfrak{a}_{\alpha\mathrm{q}}} - \rho_\alpha - \mathbb{N}\Delta_r(P_\alpha)$$

for each $s \in W$. For λ in the full open subset $\mathfrak{a}_{P_\alpha q\mathbb{C}}^{*0}(P_0, \{0\})$ of $\mathfrak{a}_{\alpha\mathrm{q}\mathbb{C}}^*$ the sets $s\lambda|_{\mathfrak{a}_{\alpha\mathrm{q}}} - \rho_\alpha - \mathbb{N}\Delta_r(P_\alpha)$ are mutually disjoint for different $[s] = W_\alpha s$ from $W_\alpha \backslash W$, see Lemmas 6.2 and 6.5. Hence

$$q_{[t],\xi}(P_\alpha, v \mid f_{\{s\}}) = 0, \tag{14.23}$$

for all $s \in W \setminus W_\alpha t$ and all $\xi \in \Delta_r(P_\alpha)$.

First assume that $W_\alpha t \cap S = \emptyset$. Then it follows from (14.23) that $q_{[t],\xi}(P_\alpha, v \mid f_S) = 0$ for all $\xi \in \Delta_r(P_\alpha)$. Hence f_S is holomorphically $[t]$-global along (P_α, v).

Next assume that $W_\alpha t \subset S$. Let $S^c = W \setminus S$. Then $f_S = f_W - f_{S^c}$, and it follows from Lemma 14.5 and what was just proved, that f_S is holomorphically $[t]$-global along (P_α, v). $\square$

If $Q \in \mathcal{P}_\sigma$ is standard, then we define the subset W^Q of W by

$$W^Q = \{s \in W \mid s(\Delta_Q) \subset \Sigma^+\} \tag{14.24}$$

It is well known, see e.g. [16], Thm. 2.5.8, that the multiplication map $W^Q \times W_Q \to W$ is bijective. Moreover, if $s \in W^Q$ and $t \in W_Q$, then $l(st) = l(s) + l(t)$; here $l\colon W \to \mathbb{N}$ denotes the length function relative to Δ. In particular this means that W^Q consists of the minimal length representatives in W of the cosets in W/W_Q.

Lemma 14.7 *Let $s \in W$, $\alpha \in \Delta$ and assume that $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$. Let $t \in W_Q$. Then $s \in W^Q t$ if and only if $s_\alpha s \in W^Q t$.*

Proof: The hypothesis $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$ is also satisfied by the elements $s_1 = st^{-1}$ and $s_2 = s_\alpha st^{-1}$. Hence we need only prove the implication $s \in W^Q \Rightarrow s_\alpha s \in W^Q$.

Assume that $s \in W^Q$. Then $s(\Delta_Q) \subset \Sigma^+$. From the hypothesis it follows that $s^{-1}\alpha \notin \Delta_Q$, hence $\alpha \notin s(\Delta_Q)$. Since α is simple, it follows that $s_\alpha(s(\Delta_Q)) \subset \Sigma^+$. Hence $s_\alpha s \in W^Q$. $\square$

Corollary 14.8 *Let $\psi \in {}^\circ\mathcal{C}[\Lambda]$ where $\Lambda \in \mathfrak{b}^*_{\mathrm{k}\mathbb{C}}$ and let $Q \in \mathcal{P}_\sigma$ be a standard parabolic subgroup. Fix $t \in W_Q$, and let the family $f\colon \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathrm{X}_+ \to V_\tau$ be defined by*

$$f(\lambda, x) = \sum_{s \in W^Q} E_{+,st}(P_0 : \lambda : x)(\psi).$$

Then $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)_{\mathrm{hglob}}$. If $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$, then the family $\mathcal{L}_* f$ belongs to the space $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{hglob}}$, where $Y = \mathrm{supp}\, \mathcal{L}$, and where δ is a suitable element of D_Q.

Proof: Let $S = W^Q t$. Then, $f = f_S$, where we have used the notation of Lemma 14.6. It follows from the mentioned lemma that $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)$. Moreover, let $s \in W$ and $\alpha \in \Delta$ be such that $s^{-1}\alpha|_{\mathfrak{a}_{Q\mathrm{q}}} \neq 0$. Then it follows from Lemma 14.7 that either $W_\alpha s \subset S$ or $W_\alpha s \cap S$ is empty. Hence it follows from Lemma 14.6 that f is holomorphically $W_\alpha s$-global along (P_α, v), for every $v \in N_K(\mathfrak{a}_\mathrm{q})$. Thus $f \in \mathcal{E}_0^{\mathrm{hyp}}(\mathrm{X}_+ : \tau : \Lambda)_{\mathrm{hglob}}$ by Lemma 13.11. The remaining assertion now follows from Theorem 13.12. $\square$

15 Asymptotics of partial Eisenstein integrals

Let $P \in \mathcal{P}_\sigma^{\min}$ and let Q be a σ-parabolic subgroup containing P. For the application of the asymptotic vanishing theorem, Theorem 9.10, in the next section we need to determine the coefficient of the leading exponent in the (Q, v)-expansion of the Eisenstein integral $E^\circ(P : \lambda)$, for every $v \in N_K(\mathfrak{a}_\mathrm{q})$. To formulate a result in this direction, we need some additional notation.

Let $v \in N_K(\mathfrak{a}_\mathrm{q})$ and select a complete set of representatives $\mathcal{W}_{Q,v}$ in $N_{K_Q}(\mathfrak{a}_\mathrm{q})$ for $W_Q/W_{K_Q \cap vHv^{-1}}$. We define ${}^\circ\mathcal{C}(Q, v) = {}^\circ\mathcal{C}(Q, v, \tau)$ to be the analogue of the space ${}^\circ\mathcal{C}$ for the data $\mathrm{X}_{1Q,v}, \tau_Q$. Thus

$$ {}^\circ\mathcal{C}(Q, v) = \oplus_{u \in \mathcal{W}_{Q,v}} C^\infty(M/M \cap uvH(uv)^{-1} : \tau) \tag{15.1}$$

with an orthogonal direct sum. Note that ${}^\circ\mathcal{C}(Q, v)$ is also the analogue of ${}^\circ\mathcal{C}$ for the data $\mathrm{X}_{Q,v}, \tau_Q$.

One readily checks that the map $\mathcal{W}_{Q,v} \to W/W_{K\cap H}$ given by $u \mapsto \mathrm{Ad}(uv)|\mathfrak{a}_\mathrm{q}$ is injective. Hence we may extend $\mathcal{W}_{Q,v} v$ to a complete set $\mathcal{W} \subset N_K(\mathfrak{a}_\mathrm{q})$ of representatives for $W/W_{K\cap H}$. If $w \in \mathcal{W}$, then $w \in \mathcal{W}_{Q,v} v \iff wv^{-1} \in K_Q$. With such choices made we have a natural isometric embedding $\mathrm{i}_{Q,v}\colon {}^\circ\mathcal{C}(Q, v) \hookrightarrow {}^\circ\mathcal{C}$, defined by

$$(\mathrm{i}_{Q,v}\psi)_w = \begin{cases} \psi_{wv^{-1}} & \text{if} \quad w \in \mathcal{W}_{Q,v} v, \\ 0 & \text{otherwise.} \end{cases} \tag{15.2}$$

The adjoint of the embedding $\mathrm{i}_{Q,v}$ is denoted by $\mathrm{pr}_{Q,v}\colon {}^\circ\mathcal{C} \to {}^\circ\mathcal{C}(Q, v)$. It is given by the following formula, for $\psi \in {}^\circ\mathcal{C}$,

$$(\mathrm{pr}_{Q,v}\psi)_u = \psi_{uv}, \qquad (u \in \mathcal{W}_{Q,v}). \tag{15.3}$$

The normalized Eisenstein integral associated with the data $\mathrm{X}_{1Q,v}, \tau_Q$ and ${}^*P{:=}\, P\cap M_{1Q}$ is denoted by $E^\circ(\mathrm{X}_{1Q,v} : {}^*P : \nu)$, for $\nu \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Similarly, the partial Eisenstein integrals associated with these data are denoted by $E_{+,s}(\mathrm{X}_{1Q,v} : {}^*P : \nu)$, for $s \in W_Q$ and $\nu \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Note that all of these are $(\tau_Q \otimes 1)$-spherical smooth functions on $\mathrm{X}_{1Q,v,+}$ with values in $\mathrm{Hom}({}^\circ\mathcal{C}(Q,v), V_\tau) \simeq V_\tau \otimes {}^\circ\mathcal{C}(Q,v)^*$.

Proposition 15.1 *Let $P \in \mathcal{P}_\sigma^{\min}$, $Q \in \mathcal{P}_\sigma$ and assume that $Q \supset P$. Let $v \in N_K(\mathfrak{a}_\mathrm{q})$, and choose $\mathcal{W}_{Q,v}$, $\mathcal{W}$ as above such that $\mathcal{W}_{Q,v} \subset \mathcal{W}v^{-1}$. Let $\psi \in {}^\circ\mathcal{C}$ and let the family $f{:}\, \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \times \mathrm{X} \to V_\tau$ be defined by*

$$f(\lambda, x) = E^\circ(P : \lambda : x)\psi.$$

Then, for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ generic, and for all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, v \,|\, f_\lambda, X, m) = E^\circ(\mathrm{X}_{1Q,v} : {}^*P : \lambda : m)\, \mathrm{pr}_{Q,v}\psi. \tag{15.4}$$

Proof: We first assume that $v = e$. Then $\mathrm{X}_{1Q,v} = \mathrm{X}_{1Q,e} = M_{1Q}/M_{1Q} \cap H$. Moreover, the set $\mathcal{W}_Q{:=}\, \mathcal{W}_{Q,e}$ is contained in $\mathcal{W}$. From [7], p. 563, Thm. 4, it follows that

$$q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, e \,|\, f_\lambda, X, m) = E^\circ(\mathrm{X}_{1Q,e} : {}^*P : \lambda : m)\, \mathrm{pr}_Q\psi,$$

for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,e,+}$. Here pr_Q is the natural projection map from ${}^\circ\mathcal{C}$ onto ${}^\circ\mathcal{C}_Q(\tau) = \oplus_{v \in \mathcal{W}_Q} C^\infty(M/M \cap vHv^{-1} : \tau_\mathrm{M})$, see [7], pp. 544 and 547. Thus, pr_Q equals the map $\mathrm{pr}_{Q,e}$ defined above and it follows that (15.4) holds with $v = e$. To establish the result for arbitrary $v \in N_K(\mathfrak{a}_\mathrm{q})$, we first need a lemma.

From Remark 3.1 we recall that $\mathrm{X}_v = \mathrm{X}_{1G,v} = G/vHv^{-1}$. The set $\mathcal{W}_{G,v}{:=}\, \mathcal{W}v^{-1}$ is a complete set of representatives for $W/W_{K\cap vHv^{-1}}$. Accordingly, the analogue ${}^\circ\mathcal{C}(G,v) = {}^\circ\mathcal{C}(G,v,\tau)$ of the space ${}^\circ\mathcal{C}$ is given by (15.1) with G in place of Q. The associated map $\mathrm{i}_{G,v}{:}\, {}^\circ\mathcal{C}(G,v) \to {}^\circ\mathcal{C}$ is now a bijective isometry; moreover, its adjoint $\mathrm{pr}_{G,v}$ is its two-sided inverse.

We recall from the end of Section 3 that right translation by v induces a topological linear isomorphism R_v from $C^\infty(\mathrm{X} : \tau)$ onto $C^\infty(\mathrm{X}_v : \tau)$. In the following lemma we will relate the right translate of $E^\circ(P : \lambda)$ to the normalized Eisenstein integral associated with X_v, $\mathcal{W}v^{-1}$ and P.

Lemma 15.2 *Let $\psi \in {}^\circ\mathcal{C}$. Then, for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$,*

$$R_v(E^\circ(\mathrm{X} : P : \lambda)\psi) = E^\circ(\mathrm{X}_v : P : \lambda)[\mathrm{pr}_{G,v}\psi]. \tag{15.5}$$

The formula remains valid if the normalized Eisenstein integrals are replaced by their unnormalized versions.

Proof: We first prove the formula for the unnormalized Eisenstein integrals. Let $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ be such that $\langle \mathrm{Re}\,\lambda + \rho_P\, ,\, \alpha\rangle < 0$ for all $\alpha \in \Sigma(P)$. Define the function

$\tilde{\psi}(\lambda)\colon G \to V_\tau$ as in [7], Eq. (19). Then $E(P : \lambda : x)\psi = \int_K \tau(k)\, \tilde{\psi}(\lambda : k^{-1}x)\, dk$. Hence $E(P : \lambda : xv)\psi = \int_K \tau(k)\tilde{\psi}_{G,v}(\lambda : k^{-1}x)\, dk$, where $\tilde{\psi}_{G,v}(\lambda : x) = \tilde{\psi}(\lambda : xv)$. One now readily checks that $\tilde{\psi}_{G,v}(\lambda)$ is the analogue of $\tilde{\psi}(\lambda)$, associated with the data $\mathrm{X}_v, \mathcal{W}v^{-1}$ and with the element $\psi_{G,v} := \mathrm{pr}_{G,v}\psi$ of ${}^\circ\mathcal{C}(G, v)$. From this we obtain the equality (15.5) for the present λ. For general λ, the result follows by meromorphic continuation.

Let $Q \in \mathcal{P}_\sigma^{\min}$. Then it follows, by application of Lemma 3.7 and the definition of the **c**-functions (cf. [7], § 4), that, for every $s \in W$, each $u \in \mathcal{W}v^{-1}$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, we have $[C_{Q|P}(\mathrm{X} : s : \lambda)\psi]_{uv} = [C_{Q|P}(\mathrm{X}_v : s : \lambda)\mathrm{pr}_{G,v}\psi]_u$. In other words,

$$\mathrm{pr}_{G,v} \circ C_{Q|P}(\mathrm{X} : s : \lambda) = C_{Q|P}(\mathrm{X}_v : s : \lambda) \circ \mathrm{pr}_{G,v}.$$

The proof is completed by combining this equation, after substitution of P and 1 for Q and s, respectively, with the unnormalized version of (15.5) and the definitions of the normalized Eisenstein integrals (cf. [7], eq. (49)). $\square$

Completion of the proof of Prop. 15.1. Let $v \in N_K(\mathfrak{a}_\mathrm{q})$ be arbitrary. Then from Lemmas 3.7, 15.2 and equation (15.4) with X_v, e and $\mathrm{pr}_{G,v}\psi$ in place of X, v and ψ, respectively, it follows that, for $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$\begin{aligned} q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, v \,|\, f_\lambda, X, m) &= q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, e \,|\, R_v(f_\lambda), X, m) \\ &= E^\circ(\tilde{\mathrm{X}}_{1Q,e} : {}^*P : \lambda : m)\, \tilde{\mathrm{pr}}_{Q,e}\mathrm{pr}_{G,v}\psi. \end{aligned}$$

In the last expression the two tildes over objects indicate that the analogues of the objects for the symmetric space X_v are taken. We now observe that $\tilde{\mathrm{X}}_{1Q,e}$ equals the space $M_{1Q}/M_{1Q} \cap evHv^{-1}e = \mathrm{X}_{1Q,v}$. Hence, to establish (15.4), it suffices to show that $\tilde{\mathrm{pr}}_{Q,e}\mathrm{pr}_{G,v}\psi = \mathrm{pr}_{Q,v}\psi$. For this we note that $\tilde{\mathrm{pr}}_{Q,e}$ is the projection from ${}^\circ\mathcal{C}(G, v)$ onto the sum of the components parametrized by the elements u of $M_{1Q} \cap \mathcal{W}v^{-1} = \mathcal{W}_{Q,v}$. Moreover, for $u \in \mathcal{W}_{Q,v}$,

$$[\tilde{\mathrm{pr}}_{Q,e}\mathrm{pr}_{G,v}\psi]_u = [\mathrm{pr}_{G,v}\psi]_u = \psi_{uv} = [\mathrm{pr}_{Q,v}\psi]_u.$$

$\square$

The result just proved generalizes to partial Eisenstein integrals.

Proposition 15.3 *Let $P \in \mathcal{P}_\sigma^{\min}$. Let $\psi \in {}^\circ\mathcal{C}$, let $S \subset W$ and let the family $f = f_S$ be defined by*

$$f(\lambda, x) = \sum_{s \in S} E_{+,s}(P : \lambda : x)\psi,$$

see Lemma 14.6. Assume that $Q \in \mathcal{P}_\sigma$ contains P and that $v \in N_K(\mathfrak{a}_\mathrm{q})$. Then, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, v \,|\, f_\lambda, X, m) = \sum_{s \in S \cap W_Q} E_{+,s}(\mathrm{X}_{1Q,v} : {}^*P : \lambda : m)\, \mathrm{pr}_{Q,v}\psi. \quad (15.6)$$

In particular, if $S \cap W_Q = \emptyset$ *then* $\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q \notin \mathrm{Exp}(Q, v \,|\, f_\lambda)$.

Proof: For $S = W$ this result is precisely Prop. 15.1. We shall use transitivity of asymptotics to derive the result for arbitrary S from it.

It suffices to prove the above identity for $m = bu \in \mathrm{X}_{Q,v,+}$, with $u \in N_{K_Q}(\mathfrak{a}_\mathrm{q})$ and $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$ arbitrary.

According to Lemma 14.3 and Remark 12.2, the function f_S belongs to $C^{\mathrm{ep}}_{0,\{0\}}(\mathrm{X}_+ : \tau : \Omega)$, for the full open subset $\Omega := \mathrm{reg}_\mathrm{a} f_S$ of $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

Hence, according to Theorem 7.8 with P_0, Q and P in place of Q, P and P_1, respectively, for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ generic the following holds, with $[1]$ the class of $1 \in W$ in $W/\sim_{Q|P_0} = W_Q\backslash W$,

$$\begin{aligned} q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, v \,|\, f_{S\lambda}, X, bu) &= q_{[1],0}(Q, v \,|\, f_S, X)(\lambda, bu) \\ &= \sum_{s \in W_Q} \sum_{\mu \in \mathbb{N}\Delta_Q(P)} b^{s\lambda - \rho_P - \mu} q_{s,\mu}(P, uv \,|\, f_S, X + \log b)(\lambda, e). \end{aligned}$$

Now, for all $s, t \in W$, $\mu \in \mathbb{N}\Delta$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$ it follows from (14.13) and Lemma 6.2 that $q_{s,\mu}(P, v \,|\, f_{\{t\}}) = 0$ if $s \neq t$. Hence

$$q_{s,\mu}(P, v \,|\, f_S) = \begin{cases} q_{s,\mu}(P, v \,|\, f_W) & \text{if} \quad s \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we obtain that

$$q_{\lambda|_{\mathfrak{a}_{Q\mathrm{q}}} - \rho_Q}(Q, v \,|\, f_{S\lambda}, X, bu) = \sum_{s \in S \cap W_Q} \sum_{\mu \in \mathbb{N}\Delta_Q(P)} b^{s\lambda - \rho - \mu} q_{s,\mu}(P, uv \,|\, f_W, X + \log b)(\lambda, e). \tag{15.7}$$

This equation is valid for any subset S of W; in particular, it is valid for $S = W$. Using (15.4) we now obtain that, for any $u \in N_{K_Q}(\mathfrak{a}_\mathrm{q})$ and all $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$,

$$E^\circ(\mathrm{X}_{1Q,v} : {}^*P : \lambda : bu)\, \mathrm{pr}_{Q,v}\psi = \sum_{s \in W_Q} \sum_{\mu \in \mathbb{N}\Delta_Q(P)} b^{s\lambda - \rho - \mu} q_{s,\mu}(P, uv \,|\, f_W, X + \log b)(\lambda, e). \tag{15.8}$$

This is the $\Delta_Q(P)$-exponential polynomial expansion of the Eisenstein integral along $({}^*P, u)$. In view of (14.12) and the remark following (14.13), with $\mathrm{X}_{1Q,v}$ in place of X, we infer from (15.8) that, for each $s \in W_Q$, and every $u \in N_{K_Q}(\mathfrak{a}_\mathrm{q})$ and $b \in {}^*A^+_{Q\mathrm{q}}({}^*P)$,

$$E_{+,s}(\mathrm{X}_{1Q,v} : {}^*P : \lambda : bu)\mathrm{pr}_{Q,v}\psi = \sum_{\mu \in \mathbb{N}\Delta_Q(P)} b^{s\lambda - \rho - \mu} q_{s,\mu}(P, uv \,|\, f_W, X + \log b)(\lambda, e). \tag{15.9}$$

Finally, (15.6) with $m = bu$ follows by combining (15.7) and (15.9). $\square$

We end this section with a generalization of Proposition 15.3, involving the application of a Laurent functional.

Proposition 15.4 *Let assumptions be as in Prop. 15.3 and let* $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$. *Then the family* $\mathcal{L}_* f$ *defined by* $\mathcal{L}_* f(\nu, x) = \mathcal{L}[f(\,\cdot\, + \nu, x)]$, *for generic* $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ *and* $x \in \mathrm{X}_+$, *belongs to* $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)$, *with* $Y = \operatorname{supp} \mathcal{L}$ *and for a suitable* $\delta \in \mathrm{D}_Q$.

Moreover, for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ *and all* $X \in \mathfrak{a}_{Q\mathrm{q}}$ *and* $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\nu - \rho_Q}(Q, v \,|\, (\mathcal{L}_* f)_\nu, X, m) = \mathcal{L}[\sum_{s \in S \cap W_Q} E_{+,s}(\mathrm{X}_{1Q,v} : {}^*P : \,\cdot\, + \nu : m) \operatorname{pr}_{Q,v} \psi]. \tag{15.10}$$

In particular, if $S \cap W_Q = \emptyset$ *then* $\nu - \rho_Q \notin \mathrm{Exp}(Q, v \,|\, (\mathcal{L}_* f)_\nu)$.

Proof: The first assertion follows from Cor. 14.4. For the second assertion, we note that $\mathcal{L}_* f \in C^{\mathrm{ep}}_{Q,Y}(\mathrm{X}_+ : \tau : \Omega)$, where Ω is the full open subset $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup \mathcal{H}_{\mathcal{L}_* f}$ of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, see Remark 12.2. The set $\Omega_* := \Omega \cap \mathfrak{a}^{*\circ}_{Q\mathrm{q}\mathbb{C}}(P, \{0\})$ is a full open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Moreover, from (7.14) it follows that, for $\nu \in \Omega_*$,

$$q_{\nu - \rho_Q}(Q, v \,|\, (\mathcal{L}_* f)_\nu, X) = q_{[1],0}(Q, v \,|\, \mathcal{L}_* f, X)(\nu), \quad (X \in \mathfrak{a}_{Q\mathrm{q}}); \tag{15.11}$$

here $[1]$ denotes the image of the identity element of W in $W/\sim_{Q|Q}$. The expression on the right-hand side of the above equation is given by (13.5), with $P = Q, \sigma = [1] \in W/\sim_{Q|Q}$ and $\xi = 0$. Note that an element $s \in W$ satisfies $[s] = [1]$ if and only if $s \in W_Q$. It follows from this that $[1] \cdot Y = \{0\}$. Hence from (13.5) and (13.2) we conclude, with $\bar{1}$ denoting the image of $1 \in W$ in $W_Q \backslash W$,

$$\begin{aligned} q_{[1],0}(Q, v \,|\, \mathcal{L}_* f, X)(\nu) &= \sum_{\lambda \in Y} \mathcal{L}^{Q,\bar{1}}_{\lambda *}[q_{\bar{1},0}(Q, v \,|\, f)(X, \,\cdot\,)](\nu, X) \\ &= \sum_{\lambda \in Y} e^{-(\lambda + \nu)(X)} \mathcal{L}_{\lambda *}[e^{(\,\cdot\,)(X)} q_{\bar{1},0}(Q, v \,|\, f)(X, \,\cdot\,)](\nu) \end{aligned} \tag{15.12}$$

for $X \in \mathfrak{a}_{Q\mathrm{q}}$ and generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. From $(\lambda + \nu)(X) = \nu(X)$ we deduce that the last expression in (15.12) equals $\sum_{\lambda \in Y} \mathcal{L}_{\lambda *}[q_{\bar{1},0}(Q, v \,|\, f)(X, \,\cdot\,)](\nu)$. Hence from (15.11) and (15.12) we obtain

$$q_{\nu - \rho_Q}(Q, v \,|\, (\mathcal{L}_* f)_\nu, X) = \mathcal{L}_*[q_{\bar{1},0}(Q, v \,|\, f)(X, \,\cdot\,)](\nu). \tag{15.13}$$

It follows from (15.6) and (7.14) that, for $X \in \mathfrak{a}_{Q\mathrm{q}}$, $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\bar{1},0}(Q, v \,|\, f)(X, \lambda, m) = \sum_{s \in S \cap W_Q} E_{+,s}(\mathrm{X}_{1Q,v} : {}^*P : \lambda : m) \operatorname{pr}_{Q,v} \psi, \tag{15.14}$$

as a meromorphic identity in $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. The equality (15.10) now follows by combining (15.13) with (15.14). $\square$

16 Induction of relations

After the preparations of the previous sections we are now able to apply the vanishing theorem, Theorem 12.10, to families obtained from applying Laurent functionals to partial Eisenstein integrals. This will lead to what we call induction of relations.

We retain the notation of the previous section. Moreover, we assume that $Q \in \mathcal{P}_\sigma$ is a standard parabolic subgroup. Thus ${}^*P_0 := M_Q \cap P_0$ is the standard minimal σ-parabolic subgroup of M_Q, relative to the positive system $\Sigma_Q^+ := \Sigma_Q \cap \Sigma$.

We assume that ${}^Q\mathcal{W}$ is a complete set of representatives in $N_K(\mathfrak{a}_\mathrm{q})$ for the double coset space $W_Q \backslash W / W_{K\cap H}$. We also assume that for each $v \in {}^Q\mathcal{W}$ a set $\mathcal{W}_{Q,v}$ as above (15.1) is chosen. Then one readily verifies that

$$\mathcal{W} = \cup_{v \in {}^Q\mathcal{W}}\ \mathcal{W}_{Q,v} v \qquad \text{(disjoint union).} \tag{16.1}$$

is a complete set of representatives for $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_\mathrm{q})$. Combining this with (15.2) and (15.3) we find that

$$\sum_{v \in {}^Q\mathcal{W}} \mathrm{i}_{Q,v} \circ \mathrm{pr}_{Q,v} = I_{{}^\circ\mathcal{C}}.$$

Combining (16.1) with (15.2) and (15.3), it also follows, for $u, v \in {}^Q\mathcal{W}$, that

$$\mathrm{pr}_{Q,u} \circ \mathrm{i}_{Q,v} = \begin{cases} I_{{}^\circ\mathcal{C}(Q,v)} & \text{if} \quad u = v; \\ 0 & \text{otherwise.} \end{cases} \tag{16.2}$$

Theorem 16.1 (Induction of relations) *Let $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$ be given for each $t \in W_Q$. If, for each $v \in {}^Q\mathcal{W}$,*

$$\sum_{t \in W_Q} \mathcal{L}_t[E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot\, : m) \circ \mathrm{pr}_{Q,v}] = 0, \qquad (m \in \mathrm{X}_{Q,v,+}) \tag{16.3}$$

then for each $s \in W^Q$ the following holds as a meromorphic identity in $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$:

$$\sum_{t \in W_Q} \mathcal{L}_t[\, E_{+,st}(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)\,] = 0, \qquad (x \in \mathrm{X}_+). \tag{16.4}$$

Conversely, if the identity (16.4) holds for some $s \in W^Q$ and all ν in a non-empty open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then (16.3) holds for each $v \in {}^Q\mathcal{W}$.

Proof: Define for each $w \in W$ the family $g_w : \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \times \mathrm{X}_+ \to V_\tau \otimes {}^\circ\mathcal{C}^*$ by

$$g_w(\nu, x) = \mathcal{L}_t[\, E_{+,st}(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)\,]$$

for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and every $x \in \mathrm{X}_+$; the elements $s \in W^Q, t \in W_Q$ are determined by the unique product decomposition $w = st$ (see below (14.24)).

It follows from Cor. 14.4, that there exist $\delta_w \in \mathrm{D}_Q$ such that $g_w \in \mathcal{E}^{\mathrm{hyp}}_{Q,Y_w}(\mathrm{X}_+ : \tau : \delta_w)$; here $Y_w = \mathrm{supp}\, \mathcal{L}_t$, where $t \in W_Q$ is determined as above. If we put $Y = \cup Y_w$ and $\delta = \max(\delta_w)$, then g_w belongs to $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)$ for all $w \in W$. Moreover, for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$,

$$\mathrm{Exp}(P_0, v \,|\, (g_w)_\nu) \subset w(\nu + Y) - \rho - \mathbb{N}\Delta. \tag{16.5}$$

In view of Proposition 15.4 it also follows for $X \in \mathfrak{a}_{Q\mathrm{q}}$, $m \in \mathrm{X}_{Q,v,+}$ and generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ that

$$q_{\nu-\rho_Q}(Q, v \,|\, (g_t)_\nu, X, m) = \mathcal{L}_t[E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot + \nu : m) \circ \mathrm{pr}_{Q,v}] \qquad (t \in W_Q), \tag{16.6}$$

and

$$q_{\nu-\rho_Q}(Q, v \,|\, (g_w)_\nu, X, m) = 0 \qquad (w \notin W_Q). \tag{16.7}$$

According to Cor. 14.8 the family $\sum_{s \in W^Q} g_{st}$ belongs to the space $\mathcal{E}^{\mathrm{hyp}}_{Q,Y}(\mathrm{X}_+ : \tau : \delta)_{\mathrm{glob}}$ for each $t \in W_Q$. Hence so does the family $g = \sum_{w \in W} g_w = \sum_{t \in W_Q, s \in W^Q} g_{st}$. Moreover, by (16.6) and (16.7)

$$q_{\nu-\rho_Q}(Q, v \,|\, (g)_\nu, X, m) = \sum_{t \in W_Q} \mathcal{L}_t[E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot + \nu : m) \circ \mathrm{pr}_{Q,v}] \qquad (m \in \mathrm{X}_{Q,v,+}).$$

From Theorem 12.10 we now see that (16.3) holds for each $v \in {}^Q\mathcal{W}$ if and only if $g = 0$.

On the other hand, let $g^s = \sum_{t \in W_Q} g_{st}$ for $s \in W^Q$. It follows from (16.5) that

$$\mathrm{Exp}(P_0, v \,|\, (g^s)_\nu) \subset s\nu + WY - \rho - \mathbb{N}\Delta.$$

Since the latter sets are mutually disjoint as s runs over W^Q, for ν in a full open subset (see Lemma 6.2), we conclude that for such ν,

$$(s\nu + WY - \rho - \mathbb{N}\Delta) \cap \mathrm{Exp}(P_0, v \,|\, g_\nu) = \mathrm{Exp}(P_0, v \,|\, (g^s)_\nu).$$

Hence $g = 0$ implies that $g^s = 0$ for each $s \in W^Q$. Conversely it follows from Corollary 9.15 that $g = 0$ if $g^s = 0$ for some $s \in W^Q$. The theorem follows immediately. $\square$

Corollary 16.2 *Let $v \in {}^Q\mathcal{W}$ and let $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}(Q, v)$ be given for each $t \in W_Q$. If*

$$\sum_{t \in W_Q} \mathcal{L}_t[E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot : m)] = 0, \qquad (m \in \mathrm{X}_{Q,v,+}) \tag{16.8}$$

then for each $s \in W^Q$ the following holds as a meromorphic identity in $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$:

$$\sum_{t \in W_Q} \mathcal{L}_t[\, E_{+,st}(\mathrm{X}\colon P_0\colon \cdot + \nu : x) \circ \mathrm{i}_{Q,v}\,] = 0, \qquad (x \in \mathrm{X}_+). \tag{16.9}$$

Conversely, if the identity (16.9) holds for some $s \in W^Q$ and all ν in a non-empty open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then (16.8) holds.

Proof: For $t \in W_Q$ we define the functional $\mathcal{L}^\circ_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$ by $\mathcal{L}^\circ_t = [I \otimes \mathrm{i}_{Q,v}](\mathcal{L}_t)$. Then for $F \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q) \otimes {}^\circ\mathcal{C}^*$ we have

$$\mathcal{L}^\circ_t F = \mathcal{L}_t[F(\,\cdot\,)\, \mathrm{i}_{Q,v}]. \tag{16.10}$$

Let $u \in {}^Q\mathcal{W}$. Then from (16.2) and (16.8) we deduce that (16.3) holds with u and $\mathcal{L}^\circ_t$ in place of v and $\mathcal{L}_t$, respectively. It follows that (16.4) holds with $\mathcal{L}^\circ_t$ in place of $\mathcal{L}_t$. In view of (16.10) this implies (16.9). The converse statement is seen similarly. □

Another useful formulation of the principle of induction of relations is the following.

Corollary 16.3 *Let $v \in {}^Q\mathcal{W}$. Let $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$ and $\varphi_t \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma) \otimes {}^\circ\mathcal{C}(Q, v)$ be given for each $t \in W_Q$. Assume that*

$$\sum_{t \in W_Q} \mathcal{L}_t[E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0\colon \cdot : m)\varphi_t(\,\cdot + \nu)] = 0, \qquad (m \in \mathrm{X}_{Q,v,+}) \tag{16.11}$$

for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Define $\psi_t = (I \otimes \mathrm{i}_{Q,v})\varphi_t \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma) \otimes {}^\circ\mathcal{C}$, for $t \in W_Q$. Then, for each $s \in W^Q$,

$$\sum_{t \in W_Q} \mathcal{L}_t[E_{+,st}(\mathrm{X}\colon P_0\colon \cdot + \nu : x)\psi_t(\,\cdot + \nu)] = 0, \qquad (x \in \mathrm{X}_+)$$

as an identity of V_τ-valued meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Proof: Let $\mathcal{H}$ be a Σ-configuration such that $\mathrm{sing}(\varphi_t) \subset \cup\mathcal{H}$, for each $t \in W_Q$. Moreover, let $Y = \cup_{t \in W_Q} \mathrm{supp}\, \mathcal{L}_t \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Fix $t \in W_Q$. Let $\mathcal{H}' := \mathcal{H}_{\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}}(Y)$ be the $\Sigma_r(Q)$-configuration in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ defined as in Corollary 11.6. Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}'$; then the function $\varphi^\nu_t\colon \lambda \mapsto \varphi_t(\lambda + \nu)$ belongs to $\mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, Y, \Sigma_Q)$. It follows from (10.7) that the functional $\mathcal{L}^\nu_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^* \otimes {}^\circ\mathcal{C}(Q, v)$ defined by

$$\mathcal{L}^\nu_t[F(\,\cdot\,)] := \mathcal{L}_t[F(\,\cdot\,)\varphi_t(\,\cdot + \nu)],$$

for $F \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q) \otimes {}^\circ\mathcal{C}(Q, v)^*$, is a ${}^\circ\mathcal{C}(Q, v)$-valued Σ_Q-Laurent functional on ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. The hypothesis (16.11) may be rewritten as (16.8) with $\mathcal{L}^\nu_t$ in place of $\mathcal{L}_t$, for each $t \in W_Q$. By application of Corollary 16.2 we therefore obtain, for $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathcal{H}'$, that

$$\sum_{t \in W_Q} \mathcal{L}_t[\, E_{+,st}(\mathrm{X}\colon P_0\colon \cdot + \mu : x)\, \psi_t(\,\cdot + \nu)\,] = 0 \tag{16.12}$$

as an identity of V_τ-valued meromorphic functions in the variable $\mu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. According to Lemma 11.9 the expression in this equation defines a meromorphic V_τ-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \times \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ whose restriction to the diagonal is a meromorphic function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Thus, if we substitute ν for μ in (16.12), we obtain an identity of V_τ-valued meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. $\square$

Corollary 16.4 *Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$. If, for each $v \in {}^Q\mathcal{W}$,*

$$\mathcal{L}_1[E_+(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot\, : m) \circ \mathrm{pr}_{Q,v}] = \mathcal{L}_2[E^\circ(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot\, : m) \circ \mathrm{pr}_{Q,v}], \qquad (m \in \mathrm{X}_{Q,v,+}) \tag{16.13}$$

then the following holds as a V_τ-valued meromorphic identity in $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$:

$$\mathcal{L}_1[\sum_{s \in W^Q} E_{+,s}(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)\,] = \mathcal{L}_2[\,E^\circ(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)\,], \qquad (x \in \mathrm{X}_+). \tag{16.14}$$

In particular, for regular values of ν, the expression on the left extends smoothly in the variable x to all of X.

Conversely, if the identity (16.14) holds for ν in a non-empty open subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, then (16.13) holds for each $v \in {}^Q\mathcal{W}$.

Proof: It follows from (14.12) that $E^\circ(\mathrm{X}_{Q,v} : {}^*P_0 : \lambda) = \sum_{t \in W_Q} E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \lambda)$. Define $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$ for $t \in W_Q$ as follows. If $t = e$ then $\mathcal{L}_t := \mathcal{L}_2 - \mathcal{L}_1$, and otherwise $\mathcal{L}_t := \mathcal{L}_2$. Then the hypothesis (16.3) in Theorem 16.1 follows from (16.13). Hence the conclusion (16.4) holds for each $s \in W^Q$. By summation over s this implies that

$$\sum_{s \in W^Q} \sum_{t \in W_Q} \mathcal{L}_t[E_{+,st}(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)] = 0, \qquad (x \in \mathrm{X}_+) \tag{16.15}$$

which, by the definition of the operators $\mathcal{L}_t$ is equivalent to (16.14).

For the converse, let $g^s(\nu, x)$ denote the expression in (16.4), as in the proof of Theorem 16.1, with $\mathcal{L}_t$ specified as above. Then it was seen in the mentioned proof that if the sum g of the g^s vanishes then so does each g^s separately. Now (16.14) implies (16.15) which exactly reads that $g = 0$. Thus (16.4) holds for each $s \in W^Q$, so that the converse statement in Theorem 16.1 can be applied. $\square$

The result just proved allows a straightforward corollary similar to Corollary 16.2, in which the maps $\mathrm{i}_{Q,v}$ are used instead of the maps $\mathrm{pr}_{Q,v}$. We omit the details. The following result is derived from Corollary 16.3 in exactly the same way as the first part of Corollary 16.4 was derived from Theorem 16.1.

Corollary 16.5 *Let $v \in {}^Q\mathcal{W}$. Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}}$ be Σ_Q-Laurent functionals on ${}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, and let $\varphi_1, \varphi_2 \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma) \otimes {}^\circ\mathcal{C}(Q, v)$. Assume that*

$$\mathcal{L}_1(E_+(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot\, : m)\varphi_1(\,\cdot\, + \nu)) = \mathcal{L}_2(E^\circ(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot\, : m)\varphi_2(\,\cdot\, + \nu)),$$

for all $m \in \mathrm{X}_{Q,v,+}$ and generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Define $\psi_j = (I \otimes \mathrm{i}_{Q,v})\varphi_j \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma) \otimes {}^\circ\mathcal{C}$, for $j = 1, 2$. Then, for every $x \in \mathrm{X}_+$,

$$\mathcal{L}_1(\sum_{s \in W^Q} E_{+,s}(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)\psi_1(\,\cdot\, + \nu)) = \mathcal{L}_2(E^\circ(\mathrm{X} : P_0 : \,\cdot\, + \nu : x)\psi_2(\,\cdot\, + \nu)),$$

as an identity of V_τ-valued meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Corollary 16.6 *Let $v \in {}^Q\mathcal{W}$ and let $\psi_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q) \otimes {}^\circ\mathcal{C}(Q, v)$ be given for each $t \in W_Q$. Let $\lambda_0 \in {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Assume that for each $m \in \mathrm{X}_{Q,v,+}$, the meromorphic V_τ-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, given by*

$$\lambda \mapsto \sum_{t \in W_Q} E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \lambda : m)\psi_t(\lambda),$$

is regular at λ_0. Then for $s \in W^Q$, $x \in \mathrm{X}_+$ and generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ the meromorphic function

$$\lambda \mapsto \sum_{t \in W_Q} E_{+,st}(\mathrm{X} : P_0 : \lambda + \nu : x)\mathrm{i}_{Q,v}\psi_t(\lambda) \tag{16.16}$$

is also regular at λ_0.

Proof: The function in (16.16) has a germ at λ_0 in $\mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \lambda_0, \Sigma_Q)$. By Lemma 10.7 it suffices to show that it is annihilated by $\mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \lambda_0, \Sigma_Q)^{*\mathcal{O}}_{\mathrm{laur}}$. Let $\mathcal{L} \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \lambda_0, \Sigma_Q)^{*\mathcal{O}}_{\mathrm{laur}}$ and define $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}, \Sigma_Q)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$ for $t \in W_Q$ by $\mathcal{L}_t = m^*_{\psi_t}\mathcal{L}$, see (10.7). The desired conclusion now follows from Corollary 16.2. $\square$

We shall now give an equivalent formulation of the induction of relations. We call it the lifting principle. For the group case a similar principle was formulated by Casselman, see [1], Thm. II.4.1, however with Eisenstein integrals that carry a different normalization.

Definition 16.7 *The space $\mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ is defined as the space of functions*

$$x \mapsto \mathcal{L}[E_+(P_0 : \,\cdot\, : x)] \in V_\tau$$

where $\mathcal{L} \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}$. It is a linear subspace of $C^\infty(\mathrm{X}_+ : \tau)$.

It follows from Corollary 14.4 with $Q = G$ that $\mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ consists of $\mathbb{D}(\mathrm{X})$-finite functions in $C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$.

Remark 16.8 Let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}$. Then $\mathcal{L}[\varphi(\,\cdot\,)E_+(P_0 : \,\cdot\,)] \in \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ for all $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)$ (see (10.7)). In particular, it follows from (14.18) that $C^\circ(s : \,\cdot\,) \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma) \otimes \mathrm{End}({}^\circ\mathcal{C})$. Hence it follows from the identity (14.14) that $\mathcal{L}[E_{+,s}(P_0 : \,\cdot\,)] \in \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ for each $s \in W$. Moreover, by similar reasoning it can be seen that the space $\mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ does not depend on the choice of $P_0 \in \mathcal{P}_\sigma^{\min}$.

Remark 16.9 Let $\lambda_0 \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma) \otimes {}^\circ\mathcal{C}$, and assume that $\lambda \mapsto E_+(P_0 : \lambda)\varphi(\lambda)$ is regular at λ_0. Then the function $x \mapsto u[E_+(P_0 : \lambda : x)\varphi(\lambda)]|_{\lambda=\lambda_0}$ belongs to $\mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ for each $u \in S(\mathfrak{a}_{\mathrm{q}}^*)$ (see the previous remark and Lemma 10.15). Moreover, it follows easily from the definition of $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^*$ that $\mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ is spanned by functions of this form.

Theorem 16.10 (Lifting principle) *Let $Q \in \mathcal{P}_\sigma$ be a standard parabolic subgroup, and let $s \in W^Q$ be fixed.*

(a) *There exists for each $v \in {}^Q\mathcal{W}$ a unique linear map*

$$F_{+,s,v} : \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_{Q,v,+} : \tau_Q) \to \mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \Sigma_r(Q), C^\infty(\mathrm{X}_+ : \tau))$$

with the following property. If $\varphi \in \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_{Q,v,+} : \tau_Q)$ is given by

$$\varphi(m) = \sum_{t \in W_Q} \mathcal{L}_t[E_{+,t}(\mathrm{X}_{Q,v} : {}^*P_0 : \,\cdot\, : m)] \qquad (m \in \mathrm{X}_{Q,v,+}), \tag{16.17}$$

for some $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \Sigma_Q)_{\mathrm{laur}}^* \otimes {}^\circ\mathcal{C}(Q, v)$, $t \in W_Q$, then

$$F_{+,s,v}(\varphi)(\nu, x) = \sum_{t \in W_Q} \mathcal{L}_t[\, E_{+,st}(\mathrm{X} : P_0 : \,\cdot + \nu : x)\, \mathrm{i}_{Q,v}\,] \tag{16.18}$$

for $x \in \mathrm{X}_+$ and generic $\nu \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$.

(b) *The function $x \mapsto F_{+,s,v}(\varphi, \nu, x)$ belongs to $\mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ for generic ν.*

(c) *The map*

$$F_{+,s} : \quad \oplus_{v \in {}^Q\mathcal{W}} \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_{Q,v,+} : \tau_Q) \to \mathcal{M}(\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*, \Sigma_r(Q), C^\infty(\mathrm{X}_+ : \tau)),$$

given by $F_{+,s}(\varphi) = \sum_v F_{+,s,v}\varphi_v$, is injective.

Proof: The uniqueness is clear from Definition 16.7. We use (16.17) and (16.18) as the definition of $F_{+,s,v}$; the fact that $F_{+,s,v}(\varphi)$ is well defined for all $\varphi \in \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_{Q,v,+} : \tau_Q)$ is equivalent with the first statement in Theorem 16.1 (see also Corollary 16.2). Once the definition makes sense, it is easily seen that $F_{+,s,v}(\varphi)$ depends linearly on φ. That $F_{+,s,v}(\varphi, \nu) \in \mathcal{A}_{\mathrm{laur}}(\mathrm{X}_+ : \tau)$ for generic ν is seen from Lemma 11.7. Finally, the injectivity of $F_{+,s}$ is equivalent with the final statement of Theorem 16.1. $\square$

Remark 16.11 Note that with $\varphi_v = E^\circ(\mathrm{X}_{Q,v} : {}^*P_0 : \lambda)$ for each $v \in {}^Q\mathcal{W}$ we obtain

$$\sum_{t \in W_Q} E_{+,st}(\mathrm{X}\colon P_0 : \lambda + \nu : x)\, \mathrm{i}_{Q,v} = F_{+,s,v}(\varphi_v, \nu, x),$$

for $x \in \mathrm{X}_+$, and hence by summation over v and s

$$E^\circ(\mathrm{X}\colon P_0 : \lambda + \nu : x) = \sum_{s \in W^Q} F_{+,s}(\varphi, \nu, x).$$

Remark 16.12 In [11], Definition 10.7, we define the generalized Eisenstein integral $E^\circ_F(\psi\colon \nu) \in C^\infty(\mathrm{X}:\tau)$ for $\psi \in \mathcal{C}_F$, $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ (with the notation of *loc. cit.*). By comparison with Theorem 16.10 for $Q = P_F$ it is easily seen that $E^\circ_F(\psi\colon \nu : x) = F_{+,1}(\psi, \nu, x)$ for $x \in \mathrm{X}_+$.

17 Appendix A: spaces of holomorphic functions

If Ω is a complex analytic manifold, then by $\mathcal{O}(\Omega)$ we denote the space of holomorphic and by $\mathcal{M}(\Omega)$ the space of meromorphic functions on Ω.

If V is a complete locally convex (Hausdorff) space, we say that a function $\varphi\colon \Omega \to V$ is holomorphic if for every $a \in \Omega$ there exists a holomorphic coordinatisation $z = (z_1, \dots, z_n)$ at a such that in a neighborhood of a the function φ is expressible as a converging V-valued power series in the coordinates z. The space of such holomorphic functions is denoted by $\mathcal{O}(\Omega, V)$. We equip this space with a locally convex topology as follows. Let $\mathcal{P}$ be a separating collection of continuous seminorms for V. For every $p \in \mathcal{P}$ and every compact set $K \subset \Omega$ we define the seminorm $\nu_{K,p}$ on $\mathcal{O}(\Omega, V)$ by $\nu_{K,p}(\varphi) = \sup_K p \circ \varphi$. This collection of seminorms is separating hence equips $\mathcal{O}(\Omega, V)$ with a locally convex topology. Note that this topology is independent of the choice of $\mathcal{P}$. Moreover, it is complete; it is Fréchet if V is a Fréchet space.

We recall that $\mathcal{O}(\Omega, V)$ is a closed subspace of $C^\infty(\Omega, V)$. Indeed, if $\bar\partial$ denotes the anti-linear part of exterior differentiation, then $\mathcal{O}(\Omega, V)$ is the kernel of $\bar\partial$ in $C^\infty(\Omega, V)$.

A densely defined function $\varphi\colon \Omega \to V$ is called meromorphic if for every $a \in \Omega$ there exists an open neighborhood U of a, and a function $\psi \in \mathcal{O}(U) \setminus \{0\}$ such that $\psi\varphi \in \mathcal{O}(U, V)$. As usual, meromorphic functions are considered to be equal if they coincide on a dense open subset. The space of V-valued meromorphic functions on Ω is denoted by $\mathcal{M}(\Omega, V)$. If φ is an V-valued meromorphic function on Ω we define $\mathrm{reg}(\varphi)$ to be the largest open subset U of Ω for which $\varphi|_U$ coincides (densely) with an element of $\mathcal{O}(U, V)$. The complement $\mathrm{sing}(\varphi) = \Omega \setminus \mathrm{reg}(\varphi)$ is called the singular locus of φ.

Lemma 17.1 *Let X be a C^∞ and Ω a complex analytic manifold. Let V be a complete locally convex space.*

Let $\mathcal{F}$ be the locally convex space of C^∞-functions $X \times \Omega \to V$ that are holomorphic in the second variable. Given $f \in \mathcal{F}$ and $x \in X$, we define the function ${}_1f(x): \Omega \to V$ by ${}_1f(x)(z) = f(x,z)$. Given $z \in \Omega$ we define the function ${}_2f(z): X \to V$ by ${}_2f(z)(x) = f(x,z)$.

(a) *The map $f \mapsto {}_1f$ defines a natural isomorphism of locally convex spaces from $\mathcal{F}$ onto $C^\infty(X, \mathcal{O}(\Omega, V))$.*

(b) *The map $f \mapsto {}_2f$ defines a natural isomorphism of locally convex spaces from $\mathcal{F}$ onto $\mathcal{O}(\Omega, C^\infty(X, V))$.*

In particular, the above maps lead to a natural isomorphism

$$C^\infty(X, \mathcal{O}(\Omega, V)) \simeq \mathcal{O}(\Omega, C^\infty(X, V)).$$

Proof: The above isomorphisms are valid with $\mathcal{O}$ replaced by C^∞ everywhere. This is a well known fact, and basically a straightforward consequence of the definitions, though somewhat tedious to check. The isomorphisms with $\mathcal{O}$ are seen to be valid by showing that the appropriate kernels of the operator $\bar{\partial}$ correspond. Checking this involves a local application of the multivariable Cauchy integral formula. $\square$

18 Appendix B: removable singularities

We discuss a variation on the idea of removable singularities for holomorphic functions that is particularly useful for application in the present paper.

A subset T of a finite dimensional complex analytic manifold Ω will be called thin if for every $\lambda \in \Omega$ there exists a connected open neighborhood U and a non-zero holomorphic function $\varphi \in \mathcal{O}(U)$ such that $T \cap U \subset \varphi^{-1}(0)$, see [21], p. 19. An open subset U of Ω will be called full if its complement is thin. It is clear that a full subset of Ω is dense in Ω. Note that the union of finitely many thin subsets is thin again; accordingly, the intersection of finitely many full open subsets of Ω is again a full open subset. Obviously any union of full open subsets is a full open subset. Note also that if Ω is connected, then every full open subset of Ω is connected ([21] p. 20).

Lemma 18.1 *Let $j: V \to W$ be an injective continuous linear map of complete locally convex Hausdorf spaces, and let F be a W-valued holomorphic function on a complex analytic manifold Ω. Assume that there exists a full open subset Ω_0 of Ω and a holomorphic function $G_0: \Omega_0 \to V$ such that such that $F = j \circ G_0$ on Ω_0. Then there exists a unique holomorphic map $G: \Omega \to V$ such that $j \circ G = F$.*

Proof: Clearly the result is of a local nature in the Ω-variable, so that we may assume that Ω is a connected open subset of $\mathbb{C}^n$, for some $n \in \mathbb{N}$. Moreover, we may as well assume that $\Omega_0 = \Omega \setminus \varphi^{-1}(0)$, with $\varphi \in \mathcal{O}(\Omega)$ a non-zero holomorphic function.

Fix $\lambda_0 \in \Omega$. Since φ is non-zero, the function $z \mapsto \varphi(\lambda_0 + z\mu)$, defined on a neighborhood of 0 in $\mathbb{C}$, is non-zero for some $\mu \in \mathbb{C}^n \setminus \{0\}$. Being holomorphic, this function then takes the value 0 in isolated points. Hence we may choose μ such that $\lambda_0 + z\mu \in \Omega_0$ for $0 < |z| \leq 1$. By compactness there exists an open neighborhood N_0 of λ_0 in Ω such that $\lambda + z\mu \in \Omega$ for all $\lambda \in N_0$ and $z \in \mathbb{C}$ with $|z| \leq 1$, and such that $\lambda + z\mu \in \Omega_0$ for $|z| = 1$. By the Cauchy integral formula we have:

$$F(\lambda) = \frac{1}{2\pi i} \int_{\partial D} F(\lambda + z\mu) \frac{dz}{z}. \tag{18.1}$$

Here ∂D denotes the boundary of the unit circle in $\mathbb{C}$, equipped with the orientation induced by the complex structure, i.e., the counter clockwise direction.

Note that the W-valued (or V-valued) integration is well defined, since W (or V) is complete locally convex. In the integrand of (18.1) the function $F(\lambda + z\mu)$ may be replaced by $j(G_0(\lambda + z\mu))$. Using that j is continuous linear we then obtain that

$$F(\lambda) = j(G(\lambda)), \tag{18.2}$$

where

$$G(\lambda) := \frac{1}{2\pi i} \int_{\partial D} G_0(\lambda + z\mu) \, \frac{dz}{z} \qquad (\lambda \in N_0).$$

Clearly $G: N_0 \to V$ is a holomorphic function; moreover, it is uniquely determined by equation (18.2), since j is injective. This implies that the local definition of G is independent of the particular choice of μ. Moreover, it also follows from (18.2) and the injectivity of j that all local definitions match and determine a global holomorphic function $G: \Omega \to V$ satisfying our requirement. $\square$

Corollary 18.2 *Let Ω_0 be a full open subset of a complex analytic manifold Ω and let X_0 be a dense open subset of a C^∞-manifold X. Moreover, let $F: \Omega \times X_0 \to \mathbb{C}$ be a C^∞ function that is holomorphic in its first variable, and assume that its restriction to $\Omega_0 \times X_0$ has a smooth extension to $\Omega_0 \times X$. Then the function F has a unique smooth extension to $\Omega \times X$. Moreover, the extension is holomorphic in its first variable.*

Proof: As in the proof of the above lemma we may as well assume that Ω is an open subset of $\mathbb{C}^n$, for some n.

Let $V = C^\infty(X)$ and $W = C^\infty(X_0)$ be equipped with the usual Fréchet topologies. Restriction to X_0 induces an injective continuous linear map $j: V \to W$.

By Lemma 17.1(b) we see that the function $\widetilde{F}: \Omega \to W$, defined by $\widetilde{F}(z) = F(z, \cdot\,)$ is holomorphic. Let G_0 be the extension of $(z, x) \mapsto F(z, x)$ to a smooth map $\Omega_0 \times X \to \mathbb{C}$. Then by density and continuity the function G_0 satisfies the Cauchy-Riemann equations in its first variable. Hence it is holomorphic

in its first variable, and it follows that the function $\widetilde{G}_0\colon \Omega_0 \to V$ defined by $\widetilde{G}_0(z) = G_0(z, \cdot\,)$ is holomorphic. From the definitions given we obtain that $\widetilde{F} = j \circ \widetilde{G}_0$ on Ω_0. By the above lemma there exists a unique holomorphic function $\widetilde{G}\colon \Omega \to V$ such that $\widetilde{F} = j \circ \widetilde{G}$. The function $G\colon (z, x) \mapsto \widetilde{G}(z)(x)$ is the desired extension of F. $\square$

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